We have discussed the motion of a massive string extensively.

This time: more examples which can be described by the wave equation.

"Longitudinal waves"

\[ \psi(x) = P_0 + \psi_p(x,t) \]

Equilibrium position

\[ P_0 : \text{room pressure} \]

\[ \rho : \text{air density} \]

\[ \Delta V \]

Change in volume:

\[ \Delta V = A \left( \psi(x + \Delta x, t) - \psi(x, t) \right) \approx A \frac{\partial \psi}{\partial x} \Delta x \]

\[ \Delta P \]

Pressure difference:

\[ -\psi_p(x + \Delta x, t) + \psi_p(x, t) \approx -\frac{\partial \psi}{\partial x} \Delta x \]

* Question: How do we relate pressure and volume?

**Ideal Gas Law**

\[(1) \quad PV = nRT \Rightarrow V \propto \frac{1}{p} = p^{-1} \]

Not quite right because this assumes that \( T \) is constant.

It is not true in sound wave.
(2) The compression is actually adiabatic meaning that there is almost NO HEAT flowing IN and OUT of the volume.

The time scale of the heat flow is larger than the time scale of oscillation.

\[ pV^\gamma = \text{const} \]

\[ pV^\gamma = C \]

Consider: small vibration

\[ \Delta p \ll P_0 \quad \Delta p: \text{change in pressure} \quad \text{w.r. } P_0 \]

\[ \Delta V \ll V_0 \]

Before: \[ P_0 V_0^\gamma = C \]

After: \[ (P_0 + \Delta P)(V_0 + \Delta V)^\gamma = C \]
\[ C = (P_0 + \Psi_p) V_0 \left( 1 + \frac{\Delta V}{V_0} \right) \]

\[ \approx (P_0 + \Psi_p) V_0 \left( 1 + \frac{\Delta V}{V_0} \right) \]

\[ \approx P_0 V_0 + \gamma \Delta V V_0 \delta p_\infty + \Psi_p V_0 + \gamma \Delta V V_0 \Psi_p \]

\[ \text{Ignore } \Delta V \Psi_p \text{ term (small)} \]

\[ \Psi_p = -\frac{\gamma P_0}{V_0} \Delta V \]

Plugin the expression we got before for

\[ \Psi_p = -\frac{\gamma P_0 A \Delta x}{V_0} \frac{\partial \Psi}{\partial x} \quad \text{(} V_0 = A \Delta x \text{)} \]

Now we know how to relate the pressure change \( \Psi_p \) and the displacement \( \Psi \)

\( \Psi(x,t) \): displacement of the air with respect to the equilibrium position \( x \)

\( \Psi_p(x,t) \): "displacement" or change in pressure with respect to the room pressure \( P_0 \)
Force acting on this volume of air:

$$ F_{\text{total}} = \Delta P \cdot A $$

$$ = A \frac{\partial \psi}{\partial x} \Delta x $$ \hspace{1cm} (from page 1)

Mass:

$$ \Delta m = \rho \cdot A \cdot \Delta x $$

\* Newton's Law

$$ F = ma $$

$$ \rho A \Delta x \ddot{y} = -A \Delta x \frac{\partial \psi}{\partial x} $$

$$ \rho \ddot{y} = \frac{\partial \psi}{\partial x} $$

$$ = \gamma \rho \frac{\partial^2 \psi}{\partial x^2} $$ \hspace{1cm} (from 3)

$$ \Rightarrow \ddot{y}(x,t) = \frac{\partial \psi}{\rho} \frac{\partial^2 \psi(x,t)}{\partial x^2} $$

Wave Equation !!!

$$ v_p = \sqrt{\frac{\partial \rho}{\rho}} $$
Adiabatic index $\gamma$

* First law of thermodynamics

$$dU + dW = dQ$$

$$U = \text{internal energy}$$

$$W = \text{work done by the sys}$$

$$Q = \text{heat supplied to the sys}$$

* Adiabatic process:

$$dU + dW = 0$$

$$W = P \, dV$$

Equipartition of Energy.

$$U = \alpha \, nRT$$

$$\alpha: \text{degrees of freedom}$$

$$= \alpha \, PV$$

$$dU = \alpha \left( \frac{dP}{P} \right) V + P \, dV$$

$$= -SW = -P \, dV$$

$$(\alpha + 1) \, P \, dV = -\alpha \, V \, dP$$

$$\frac{dP}{P} = -\left( \frac{\alpha + 1}{\alpha} \right) \frac{dV}{V} = -\gamma \frac{dV}{V}$$

$$\gamma = \frac{\alpha + 1}{\alpha}$$

$$\Rightarrow PV^\gamma = \text{const}$$

monoatomic gas

$$\alpha = \frac{5}{2}$$

$$\gamma = \frac{7}{5}$$

diatomic gas

3 translational degrees of freedom

Not excited until high
$\gamma$ for diatomic gas is 7/5

Air at sea level: $P_0 \approx 10^5 \text{ kg/ms}^2$

Air density: $\rho = 1.2 \text{ kg/m}^3$

$\Rightarrow$ Speed of sound: $v_p = 342 \text{ m/s}$

Experiment: $v_p = 343 \text{ m/s}$ at 70 F

Very nice agreement!!

What have we learned?

(A) The speed of sound increases

if we use monatomic gas

to replace diatomic gas (Air)

$\gamma \Rightarrow$ increases

If wave length is fixed

$\Rightarrow$ higher frequency
Search for 4 nodes \( \Rightarrow 4\lambda \)

\[ \Rightarrow \lambda = \frac{\nu}{f} = \frac{360 \, \text{m/s}}{1000} = 0.36 \, \text{m} \]

\[ \Rightarrow \nu = f \cdot \lambda \]

\[ \frac{3}{2} \text{cm} = \frac{1}{4} \text{cm} \]

\[ f = 1 \, \text{kHz} \]

\[ \Rightarrow \nu = 34 \, \text{cm} / 1 \, \text{kHz} \]

\[ = 34 \times 10^3 \, \text{cm/s} \]

\[ = 340 \, \text{m/s} \]

Very close to the calculated result in page 6!!! (342 m/s)

Rule out the prediction from Newton!
(B) The fact that they are described by wave eq:

(1) \[ W = U_p K \]

(2) Normal modes:
\[ \psi(x) = \sum_{m=1}^{\infty} A_m \sin(k_m x + \alpha_m) \sin(w_m t + \beta_m) \]

(3) \( k_m, \alpha_m \): determined by boundary conditions

(4) \( A_m, \beta_m \): determined by initial conditions.

Example:

Audio analyzer recorded the following energy vs frequency:

![Graph](image)

\( E \)

1000 2000 3000 4000 5000

Frequency (Hertz)

(i) Which configuration gave rise to this power spectrum?

(A) \[ L \]

(B) \[ L \]

(C) \[ L \]

(D) \[ L \]

→
Boundary condition for a closed end:

$$\psi = 0 \quad : \quad \text{air can go nowhere :)}$$

Boundary condition for an open end

$$\frac{\partial \psi}{\partial z} = 0 \quad : \quad \text{Pressure = room pressure.}$$

(A) \hspace{1cm} 0 \hspace{1cm} \Theta

$$\psi(0) = 0 \quad , \quad \psi(L) = 0 \quad \Rightarrow \quad \psi_m = \frac{A_m \sin (k_m x + \alpha_m)}{\sin (k_m L + \beta_m)}$$

1. \hspace{1cm} \Rightarrow \quad \sin (\alpha_m) = 0
   \hspace{1cm} \Rightarrow \quad \alpha_m = 0

2. \hspace{1cm} \Rightarrow \quad \sin (k_m L) = 0 \quad \Rightarrow \quad k_m = \frac{m \pi}{L}
   \hspace{1cm} \Rightarrow \quad \omega_m = \frac{m \pi v}{L}

(B) \hspace{1cm} 0 \hspace{1cm} \Theta

$$\psi(0) = 0 \quad \frac{\partial \psi(L)}{\partial z} = 0 \quad \Rightarrow \quad \alpha_m = 0$$

1. \hspace{1cm} \Rightarrow \quad \cos (k_m x) = 0 \quad \Rightarrow \quad k_m = \frac{\left(m - \frac{1}{2}\right) \pi}{L}

(C) \hspace{1cm} 0 \hspace{1cm} \Theta

$$\frac{\partial^2 \psi(0)}{\partial z^2} = 0 \quad \Rightarrow \quad \omega_m = \frac{(m - \frac{1}{2}) \pi v}{L} \Rightarrow \text{Match the data!}$$

1. \hspace{1cm} \Rightarrow \quad \cos (\alpha_m) = 0 \quad \Rightarrow \quad \alpha_m = \frac{\pi}{2}

2. \hspace{1cm} \Rightarrow \quad \sin (k_m L + \frac{\pi}{2}) = 0 \quad \Rightarrow \quad k_m = \frac{\frac{m \pi}{L}}{L} \quad \omega_m = \frac{m \pi v}{L}$
(ii) Normal modes:  

<table>
<thead>
<tr>
<th>m=1</th>
<th>m=2</th>
<th>m=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Pattern A]</td>
<td>![Pattern B]</td>
<td>![Pattern C]</td>
</tr>
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</table>

Amplitude:  

<table>
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<tr>
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<td>![Amplitude B]</td>
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</table>

(iii) How long does it take for this amplitude pattern to reappear? \[ \frac{2\pi}{\omega} \]

(iv) What about pressure? 

in each normal mode?  

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\[ \Psi_p = \psi_0 \frac{\partial \psi}{\partial x} \]

(Opposite Pattern v.s. Amplitude.)
1) Drive the organ

\[ \leftrightarrow \quad D \cos (W_d t) \]

\[ \psi (0) = 0 , \quad \psi (L) = D \cos (W_d t) \]

\[ \lambda = \frac{W_d}{2} \] (decided by the dispersion relation)

\[ \psi (x) = A_d \sin (\lambda x + \alpha) \cos (W_d t) \]

1) \( \psi (0) = 0 \quad \Rightarrow \quad \alpha = 0 \)

2) \( \psi (L) = D \cos (W_d t) \)

\[ \Rightarrow \quad A_d \sin (\lambda L) = D \]

\[ A_d = \frac{D}{\sin (\lambda L)} \]

\[ \Rightarrow \quad \psi (x) = \frac{D}{\sin (\lambda L)} \sin (\lambda x) \cos (W_d t) \]

When \( \lambda = \frac{(m + \frac{1}{2}) \pi}{L} \) \( \Rightarrow \) resonance!

( Huge amplitude )