* Review of Lecture 1

Another Example:

**LC circuit**

![LC circuit diagram]

\[ V = 0 \]

\[ I(t) = \frac{d Q(t)}{d t} \]

At \( t = 0 \), \( I(0) = I_{\text{initial}}, \ Q(0) = 0 \)

"Initial Condition"

* Voltage Drop:

\[ L \frac{dI}{dt} \]

Capacitor:

\[ V = \frac{Q}{C} \]

\[ L \frac{dI}{dt} + \frac{Q}{C} = 0 \Rightarrow \ddot{Q} + \frac{Q}{LC} = 0 \]

\[ \omega_0 = \sqrt{\frac{1}{LC}} \]

Initial Condition

\( Q(t) = A \cos(\omega_0 t + \phi) \)

\( M \leftrightarrow L \)

\( Q(t) = \frac{I_{\text{initial}}}{\omega_0} \sin(\omega_0 t) \)

\( k \leftrightarrow \frac{1}{C} \)

(From \( I(0) \) and \( Q(0) \) we can solve and get)

\( \phi = -\frac{\pi}{2}, \ A = \frac{I_{\text{initial}}}{\omega_0} \)
\[ M \frac{d^2x}{dt^2} = -kx \]

**Kinetic Energy** = \[ \frac{1}{2} M \left( \frac{dx}{dt} \right)^2 \]

**Potential Energy** = \[ \frac{1}{2} kx^2 \]

**Total Energy** \[ E = \frac{1}{2} M \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} kx^2 \]

If we solve the equation:

\[ \omega_0 = \sqrt{\frac{k}{M}} \quad \ddot{x} + \omega_0^2 x = 0 \]

\[ x(t) = A \cos(\omega_0 t + \phi) \]

\[ \frac{dx(t)}{dt} = -A \omega_0 \sin(\omega_0 t + \phi) \]

\[ E = \frac{1}{2} M A^2 \omega_0^2 \sin^2(\omega_0 t + \phi) + \frac{1}{2} kA^2 \cos^2(\omega_0 t + \phi) \]

\[ = \frac{1}{2} kA^2 \left( \sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi) \right) \]

\[ = \frac{1}{2} kA^2 \quad \text{Constant} \quad \text{!!!} \]

\[ \Downarrow \text{proportional to} \quad A^2 \quad \text{amplitude} \]

\[ \Downarrow \text{proportional to} \quad k \quad \text{"spring constant"} \]

\[ E \]

\[ \text{Kinetic E.} \]

\[ \text{Potential E.} \]
Let's look at this example:

**Newton's Law** \( \tau = I \alpha \)

*Origin*: \( \Theta = 0 \) ⇒ Pointing downward

*Define Anti-clockwise Rotation to be Positive*

*Initial Condition*: At \( t = 0 \) ⇒ \( \begin{cases} \Theta(0) = \Theta_{\text{INITIAL}} \\ \dot{\Theta}(0) = 0 \end{cases} \)

**Force Diagram:**

\[ \tau = \vec{R} \times \vec{F} \]

\[ \tau = -mg \frac{d}{2} \sin \Theta(t) \]

**Newton's Law:** \( I \alpha(t) = I \ddot{\Theta}(t) = -mg \frac{d}{2} \sin \Theta(t) \)

\[ I \ddot{\Theta} = -mg \frac{d}{2} \sin \Theta(t) \]

\[ I = \frac{1}{3} ml^2 \]

\[ \Rightarrow \ddot{\Theta}(t) = -3g \frac{d}{2l} \sin \Theta(t) = -W_0^2 \sin \Theta(t) \]

\[ W_0 = \sqrt{\frac{3g}{2l}} \]

Now again: We have translated the physical situation to mathematics. This contains everything we know.

We need to solve this equation.

However, life is hard!

We don't know how to solve \( \ddot{\Theta} = -W_0^2 \sin \Theta \)

Not the end of world, we can solve it by a computer or ....
We can consider a special case: Small angle limit

\[ \Theta(t) \to 0 \implies \sin \Theta(t) \approx \chi \]

\[ \Theta = 1^\circ \implies \frac{\sin \Theta}{\Theta} = 99.99\% \]
\[ 5^\circ \implies 99.9\% \]
\[ 10^\circ \implies 99.5\% \]

The approximation is quite good!

Then the equation of motion becomes:

\[ \ddot{\Theta}(t) = -\omega_0^2 \Theta(t) \quad \omega_0 = \sqrt{\frac{3g}{2l}} \]

We have solved this in previous lectures!

\[ \Theta(t) = A \cos (\omega_0 t + \phi) \]

Initial conditions:
We conclude \( 0 = -\omega A \sin \phi \implies \phi = 0 \)
\( \Theta_{\text{INITIAL}} = A \)
\[ \therefore \Theta(t) = \Theta_{\text{INITIAL}} \cos (\omega_0 t) \]
\[ \omega_0 = \sqrt{\frac{3g}{2l}} \]

In case if you have not noticed:

\[ \omega_0 = \sqrt{\frac{g}{l}} \]

\( \text{All those systems} \)
Now we will add a drag force:

\[ \tau_{\text{drag}}(t) = -b \dot{\theta}(t) \]

We choose this form: not because it is the most realistic description, but because this is solvable.

(Also not bad at all!)

If we choose another form of drag force:

\[ \Rightarrow \text{have to solve it by computer} \]

Now teaching physics \( \Rightarrow \) That’s why we use those approximation + assumption in class.

* EQUATION OF MOTION:

\[ \ddot{\theta}(t) = \frac{\tau(t)}{I} = \frac{\tau_g(t) + \tau_{\text{drag}}(t)}{I} \]

\[ \approx -mg \frac{L}{2} \sin\theta(t) - b \dot{\theta}(t) \]

Small angle

\[ \tau = \frac{3g}{2} \theta(t) - \frac{3b}{ml^2} \dot{\theta}(t) \]

Define \( \omega_0^2 = \frac{3g}{2} \quad \tau = \frac{3b}{ml^2} \)

Again: The reason we define \( \omega_0 \) and \( \tau \) is to simplify things, to make our life easier.

\[ \Rightarrow \dot{\theta}(t) + \tau \dot{\theta}(t) + \omega_0^2 \theta(t) = 0 \]

Oscillation Frequency

Answer: \( W \geq \omega_0 \text{ or } W \leq \omega_0 \)
Now we want to solve this equation.

\[ \Theta(t) + \Gamma \dot{\Theta}(t) + \omega_0^2 \Theta(t) = 0 \]

Use complex notation!

\[ \Theta(t) = \text{Re} \left( z(t) \right) \quad z(t) = e^{i\alpha t} \]

\[ \Rightarrow \quad \ddot{z}(t) + \Gamma \dot{z}(t) + \omega_0^2 z(t) = 0 \]

\[ \left( -\alpha^2 + i\Gamma \alpha + \omega_0^2 \right) e^{i\alpha t} = 0 \]

\[ e^{i\alpha t} \text{ is never } 0. \]

\[ \Rightarrow \quad \alpha^2 - i\Gamma \alpha - \omega_0^2 = 0 \]

\[ \Rightarrow \quad \alpha = \frac{i\Gamma \pm \sqrt{4\omega_0^2 - \Gamma^2}}{2} = \frac{i\Gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}} \]

1. If \( \omega_0^2 > \frac{\Gamma^2}{4} \) "Underdamped Oscillators"

   The drag force is small

\[ \Rightarrow \text{Define } \omega^2 = \omega_0^2 - \frac{\Gamma^2}{4} \]

\[ z_+(t) = e^{-\frac{\Gamma}{2} t} e^{i\omega t} \]

\[ z_-(t) = e^{-\frac{\Gamma}{2} t} e^{-i\omega t} \]

\[ \text{Ans: Slower.} \]
\[ \Theta_1(t) = \frac{1}{2}(Z_+(t) + Z_-(t)) \]
\[ = e^{-\frac{\Gamma}{2}t} \cos \omega t \]
\[ \Theta_2(t) = \frac{-i}{2} (Z_+(t) - Z_-(t)) \]
\[ = e^{-\frac{\Gamma}{2}t} \sin \omega t \]
\[ \Theta(t) = e^{-\frac{\Gamma}{2}t} \left[ a \cos \omega t + b \sin \omega t \right] \]
or
\[ \Theta(t) = Ae^{-\frac{\Gamma}{2}t} \left[ \cos (\omega t + \phi) \right] \]

Use the initial condition:

\[ \Theta(0) = \Theta_{\text{INITIAL}} = A \cos \phi \]
\[ \dot{\Theta}(0) = -\frac{Ar}{2} \cos \phi - Aw \sin \phi = 0 \]

We can solve \( A \) and \( \phi \)

\[ \tan \phi = -\frac{1}{2w} \quad \phi = \tan^{-1} \left( -\frac{1}{2w} \right) \]

\[ A = \frac{\Theta_{\text{INITIAL}}}{\cos \phi} \]

\( \Theta(t) \) amplitude

\[ e^{-\frac{\Gamma}{2}t} \]

Diagram showing oscillatory behavior with decay for \( t \to \frac{2\pi}{\omega} \), \( \not= W_0 \) for \( W < W_0 \).
2. \[ \omega_0^2 = \frac{I^2}{4} \] "Critically Damped Oscillator"

This means that \( \omega = 0 \)!

Starting from 1.

\[ \Theta(t) = e^{-\frac{I}{2}t} \cos\omega t \quad \overset{\omega \to 0}{\longrightarrow} \quad e^{-\frac{I}{2}t} \]

\[ \Theta_2(t) = e^{-\frac{I}{2}t} \sin\omega t \quad \overset{\omega \to 0}{\longrightarrow} \quad 0 \]

So instead... we do:

\[ \frac{\Theta_2(t)}{\omega} = \frac{1}{\omega} e^{-\frac{I}{2}t} \sin\omega t \quad \overset{\omega \to 0}{\longrightarrow} \quad t e^{-\frac{I}{2}t} \]

\[ \Rightarrow \text{Linear combination of } \Theta_1(t) \text{ and } \frac{\Theta_2(t)}{\omega} : \]

\[ \Theta(t) = (A + Bt) e^{-\frac{I}{2}t} \]

Prediction: No oscillation!

Application: √ Door close

√ Suspension System
3. \( w^2 < \frac{\Gamma^2}{4} \)  

Huge drag force!

\[
\alpha = \frac{i \Gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}
\]

"Overdamped Oscillator"

\[
= i \left( \frac{\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \omega_0^2} \right)
\]

Define \( \Gamma_\pm = \frac{\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \omega_0^2} \)

\[
\Rightarrow \text{Solution: } \Theta(t) = A_+ e^{-\Gamma_+ t} + A_- e^{-\Gamma_- t}
\]

No oscillation! Two exponentially decaying terms.