We consider the highly idealized system:

Where neither block is initially moving, but the second block is displaced at a small angle at \( t = 0 \). There is no drag force, the springs are ideal. We want to predict the motion at arbitrary times. 

Define the coordinate system where \( \vec{x}_1 \) and \( \vec{x}_2 \) are measured from the equilibrium position. The \( \hat{x} \) direction is to the right and the \( \hat{y} \) direction is up.

Implementing the small angle approximation: \( \Rightarrow \cos \theta_1 \approx 1 \), \( \sin \theta_1 \approx \theta_1 \)

From the \( \hat{y} \) direction we get \( T_1 = mg \)

\[
m\ddot{y}_1 = T_1 \cos \theta_1 - mg
\]

\[
\hat{x} \text{ direction:} \\
m\ddot{x}_1 = -T_1 \sin \theta_1 + k(x_2 - x_1)
\]

Similarly \( m\ddot{x}_2 = kx_1 - \left( k + \frac{mg}{l} \right) x_2 \)
Convert everything to matrix form (recall $M\ddot{X} = -KX$)

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad K = \begin{pmatrix} k + mg/l & -k \\ -k & k + mg/l \end{pmatrix} \quad M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$M^{-1}K = \begin{pmatrix} k/m + g/l & -k/m \\ -k/m & k/m + g/l \end{pmatrix}$$

Our equation of motion is $\ddot{X} = -M^{-1}KX$. We need to solve the eigenvalue problem. This is easiest if we switch to complex notation, define: $X = \text{Re}[Z]$ and $Z = e^{i(\omega t + \phi)}A$. The equation of motion becomes

$$\omega^2 A = M^{-1}K A$$

and we need to solve

$$\det(M^{-1}K - \omega^2 I)A = 0$$

$$M^{-1}K - \omega^2 I = \begin{pmatrix} g/l + k/m - \omega^2 & -k/m \\ -k/m & g/l + k/m - \omega^2 \end{pmatrix}$$

$$(g/l + k/m - \omega^2)^2 - (k/m)^2 = 0$$

$$(g/l + k/m - \omega^2) = \pm (k/m)$$

$$\Rightarrow \omega^2 = \frac{g}{l}, \frac{g}{l} + \frac{2k}{m}$$

Where we define $\omega_1^2$ as the first and $\omega_2^2$ as the second solution.

First examine 1: $\omega_1^2 = \frac{g}{l}$

$$(M^{-1}K - \omega_1^2 I)A = \begin{pmatrix} k/m & -k/m \\ -k/m & k/m \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \Rightarrow A^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Next examine 2: $\omega_2^2 = \frac{g}{l} + \frac{2k}{m}$

$$(M^{-1}K - \omega_2^2 I)A = \begin{pmatrix} -k/m & -k/m \\ -k/m & -k/m \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \Rightarrow A^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Go back to $X$: $X = \text{Re}[Z] = \text{Re}[e^{i(\omega t + \phi)}A]$

$$X^{(1)} = \cos(\omega_1 t + \phi_1)A^{(1)}$$

$$X^{(2)} = \cos(\omega_2 t + \phi_2)A^{(2)}$$

Where $\omega_1 \equiv \sqrt{g/l}$ and $\omega_2 \equiv \sqrt{g/l + 2k/m}$ as above. The full solution is then:

$$x_1 = \alpha \cos(\omega_1 t + \phi_1) + \beta \cos(\omega_2 t + \phi_2)$$

$$x_2 = \alpha \cos(\omega_1 t + \phi_1) - \beta \cos(\omega_2 t + \phi_2)$$

Where the initial conditions can be used to determine $\alpha, \beta, \phi_1, \phi_2$. Implementing our initial conditions from above we find:

$$\alpha = x_0/2 \quad \beta = -x_0/2 \quad \phi_1 = \phi_2 = 0$$
Rewriting our full solution:

\[ x_1 = \frac{x_0}{2} (\cos \omega_1 t - \cos \omega_2 t) \]
\[ x_2 = \frac{x_0}{2} (\cos \omega_1 t + \cos \omega_2 t) \]

Or if we implement some trig identities:

\[ x_1 = -x_0 \sin \left( \frac{\omega_1 + \omega_2}{2} t \right) \sin \left( \frac{\omega_1 - \omega_2}{2} t \right) \]
\[ x_2 = x_0 \cos \left( \frac{\omega_1 + \omega_2}{2} t \right) \cos \left( \frac{\omega_1 - \omega_2}{2} t \right) \]

If \( \omega_1 \approx \omega_2 \) (e.g. \( \omega_1 = 0.9 \omega_2 \))

\[ \frac{\omega_1 + \omega_2}{2} = 0.95 \omega_2 \]
\[ \frac{\omega_1 - \omega_2}{2} = -0.05 \omega_2 \]

We get two distinct waves: a carrier (high frequency) and the “beat” (low frequency) with the periods as shown.

We can define a “normal coordinate:” \( U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \equiv \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix} \)

\[ U_1 = 2A \cos (\omega_A t + \phi_1) \]
\[ U_2 = 2B \cos (\omega_B t + \phi_2) \]

\[ m(\ddot{x}_1 + \ddot{x}_2) = -\left( \frac{mg}{l} \right) (x_1 + x_2) \]
\[ m(\ddot{x}_1 - \ddot{x}_2) = -\left( \frac{mg}{l} + 2k \right) (x_1 - x_2) \]
We’ve successfully decoupled the equations of motion!

\[ m\ddot{U}_1 = -\left(\frac{mg}{l}\right) U_1 \]
\[ m\ddot{U}_2 = -\left(\frac{mg}{l} + 2k\right) U_2 \]

Where \( U_1 \) (and \( U_2 \)) are oscillating harmonically at \( \omega_1 \) (and \( \omega_2 \))!!!