Examples of coupled oscillations:

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Next we will look at driven coupled oscillators.

Last time:
We solved the normal mode of this system. Now we would like to add a driving force on left mass.

\[ F_d = F_0 \cos(\omega dt) \hat{x} \]

Equations of motion:

\[
\begin{align*}
    m\ddot{x}_1 &= -\left( k + \frac{mg}{l} \right) x_1 + kx_2 + F_0 \cos(\omega dt) \\
    m\ddot{x}_2 &= kx_1 - \left( k + \frac{mg}{l} \right) x_2
\end{align*}
\]

Putting the equation of motion into matrix form we have:

\[
M\ddot{X} = -KX + F\cos(\omega dt)
\]

where

\[
M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad K = \begin{pmatrix} k + \frac{mg}{l} & -k \\ -k & k + \frac{mg}{l} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

\[
\Rightarrow \ddot{X} = -M^{-1}KX + M^{-1}F\cos(\omega dt)
\]
\[ M^{-1}K = \begin{pmatrix} \frac{k}{m} + \frac{q}{l} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} + \frac{q}{l} \end{pmatrix} \quad M^{-1}F = \begin{pmatrix} F_0 \\ 0 \end{pmatrix} \]

Last time we solved the homogeneous equation:

\[ \det(M^{-1}K - \omega^2I) = 0 \]

Recall the solutions:

\[ \omega_1^2 = \frac{g}{l} \quad A^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \omega_2^2 = \frac{g}{l} + \frac{2k}{m} \quad A^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

\[ \det(M^{-1}K - \omega^2I) = (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) = 0 \]

Homogeneous solution:

\[ x = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1t + \phi_1) + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2t + \phi_2) \]

Now we have an additional driving force:

\[ \ddot{X} + M^{-1}KX = M^{-1}F \cos(\omega_d t) \]

Similar to driven oscillator problem, we want to eliminate the \( \cos(\omega_d t) \) term...

Go to complex notation:

\[ X = \text{Re}[Z] \quad \ddot{Z} + M^{-1}KZ = M^{-1}Fe^{i\omega_d t} \]

Guess: \( Z = Be^{i\omega_d t} \) where \( B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \)

Plug our guess for \( Z \) into the equation:

\[ \Rightarrow (M^{-1}K - \omega_d^2I)Be^{i\omega_d t} = M^{-1}Fe^{i\omega_d t} \]

\[ \Rightarrow (M^{-1}K - \omega_d^2I)B = M^{-1}F \]

These are just two simultaneous equations:

\[ \begin{pmatrix} \frac{k}{m} + \frac{q}{l} - \omega_d^2 \\ -\frac{k}{m} & \frac{k}{m} + \frac{q}{l} - \omega_d^2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} F_0 \\ 0 \end{pmatrix} \]

\[ \begin{pmatrix} \frac{k}{m} + \frac{q}{l} - \omega_d^2 \\ -\frac{k}{m} \end{pmatrix} B_1 - \frac{k}{m}B_2 = \frac{F_0}{m} \]

\[ \begin{pmatrix} \frac{k}{m} + \frac{q}{l} - \omega_d^2 \\ -\frac{k}{m} \end{pmatrix} B_1 - \frac{k}{m}B_2 = 0 \]

We can go ahead and solve it directly to get \( B_1 \) and \( B_2 \) or we can use “Cramer’s Rule” which is a useful rule when solving a large number of coupled oscillators.

First define:

\[ \vec{E} = \begin{pmatrix} \frac{k}{m} + \frac{q}{l} - \omega_d^2 \\ -\frac{k}{m} & \frac{k}{m} + \frac{q}{l} - \omega_d^2 \end{pmatrix} \quad \vec{D} = \begin{pmatrix} F_0 \\ 0 \end{pmatrix} \]
To use Cramer’s rule, use one column from $\vec{E}$ and $\vec{D}$

$$B_1 = \frac{|(\vec{D})()|}{\det \vec{E}}$$

$$= \frac{\left( \frac{F_0}{m} \quad (\frac{k}{m} + \frac{g}{l} - \omega_d^2) \right)}{(\omega_d^2 - \omega_1^2)(\omega_d^2 - \omega_2^2)}$$

$$= \frac{\left( \frac{k}{m} \quad (\frac{k}{m} + \frac{g}{l} - \omega_1^2) \right)}{(\omega_d^2 - \omega_1^2)(\omega_d^2 - \omega_2^2)}$$

Which explodes when $\omega_d = \omega_1, \omega_2$ which are the frequencies of the normal modes. Similarly:

$$B_2 = \frac{|()(\vec{D})|}{\det \vec{E}}$$

$$= \frac{\left( \frac{k}{m} + \frac{g}{l} - \omega_d^2 \quad \frac{F_0}{m} \right)}{(\omega_d^2 - \omega_1^2)(\omega_d^2 - \omega_2^2)}$$

$$= \frac{\left( \frac{k}{m} \right)}{(\omega_d^2 - \omega_1^2)(\omega_d^2 - \omega_2^2)}$$

$$B_1 \cdot B_2 = \frac{k/m + g/l - \omega_d^2}{k/m}$$

$$B_1 = \frac{\omega_d^2 = \omega_1^2}{\omega_d^2 = \omega_2^2} = \frac{g}{k} \Rightarrow \frac{B_1}{B_2} = 1$$

$$B_2 = \frac{\omega_d^2 = \omega_2^2}{\omega_d^2 = \omega_1^2} = \frac{g}{k} + \frac{2k}{m} \Rightarrow \frac{B_1}{B_2} = -1$$

Full solution:

$$x_1 = \alpha \cos(\omega_1 t + \phi_1) + \beta \cos(\omega_2 t + \phi_2) + B_1 \cos(\omega_d t)$$

$$x_2 = \alpha \cos(\omega_1 t + \phi_1) - \beta \cos(\omega_2 t + \phi_2) + B_2 \cos(\omega_d t)$$

Where the term with $B$ amplitude is the particular solution and the terms with $\alpha$ and $\beta$ amplitude are the homogeneous solution.
$B_1 \sim B_2 \quad B_1 \sim -B_2$

↑  ↑

Excite Mode 1  Excite Mode 2
(Near $\omega_1$)  (Near $\omega_2$)