Reminder: EXAM Review section
This is what we have done:

From \( \frac{a}{\cdots} \) to \( \frac{a}{\cdots} \)
\( N \)-coupled oscillator \( \infty \) coupled oscillator

\( \Rightarrow \) \( N \) coupled equations of motion \( \Rightarrow \infty \) coupled equation of motion

\( \Box \) Idea we got: make use of the property:

"Space Translation Invariance"

This symmetry can be translated into mathematics

\[ A' = SA \text{ such that } A_j' = A_{j+i} \]

If \( A \) is an eigenvector of \( S \)

\( \Rightarrow SA = \beta A \)

\( \Rightarrow A_j' = \beta A_j = A_{j+i} \)

\( \Rightarrow A_j = \beta^j A_0 \propto \beta^j \)

Consider \( \beta = e^{ikx} \) (don't want \( A_j \rightarrow \infty \) when \( j \rightarrow \infty \)).

\( \Rightarrow A_j \propto e^{ijkx} \)

Need \( |\beta| = 1 \)

\( \beta = e^{i\theta} \)

factorize the length scale \( a \) out
\( (a: \text{ space between masses}) \)
Let's consider this example:

A lot of point-like massive particles connected by massless strings.

These particles can only move up and down. We have constant tension $T$ and small vibration. Distance between particles: $A$

**Question:** what will be the resulting motion of the system?

**Force diagram:**

Assume $Y_j \ll A \Rightarrow \theta_1, \theta_2 \ll 1$

**Horizontal Direction:**

$$M \ddot{x}_j = -T \cos \theta_1 + T \cos \theta_2$$  \hspace{1cm} \text{(1)}

**Vertical Direction:**

$$M \ddot{y}_j = -T \sin \theta_1 - T \sin \theta_2$$  \hspace{1cm} \text{(2)}

Since $\theta_1$ and $\theta_2$ are very small $\Rightarrow \cos \theta \approx 1$, $\sin \theta \approx \theta$

1. $\Rightarrow \ M \ddot{x}_j = -T + T = 0$ No motion in the horizontal direction.

2. $\Rightarrow \ M \ddot{y}_j = -T (\sin \theta_1 + \sin \theta_2)$

$$\dot{\ddot{y}_j} \approx -T \left( \frac{Y_j - Y_{j-1}}{a} + \frac{Y_j - Y_{j+1}}{a} \right)$$

$$M \ddot{y}_j = \frac{T}{a} \left( Y_{j-1} - 2Y_j + Y_{j+1} \right)$$
Normal modes: \( y_j = \text{Re} \left( A_j e^{i(\omega t + \phi)} \right) \)

From \( S \) matrix, the eigenvectors are \( A = \begin{pmatrix} A_j \\ A_{j-1} \\ A_{j+1} \end{pmatrix} \)

\( A_j \propto \beta_j = e^{ijk a} \)

* reminder: \( a \) distance between particles in the \( x \) direction

To get \( M^T K \) matrix:

\[
M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad K = \begin{pmatrix} -\frac{1}{a} & \frac{2}{a} & -\frac{1}{a} & \ldots & 0 \\ 0 & -\frac{1}{a} & \frac{2}{a} & -\frac{1}{a} & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & -\frac{1}{a} \end{pmatrix}
\]

\( M^{-1} K = \begin{pmatrix} \cdots & \frac{-1}{ma} & \frac{2}{ma} & \frac{-1}{ma} & \cdots \\ \cdots & 0 & \frac{-1}{ma} & \frac{2}{ma} & \frac{-1}{ma} \end{pmatrix} \)

To get \( \omega \), since \( M^T K \) and \( S \) share the same eigenvectors

Calculate \( M^{-1} K A = \omega^2 A \)

\( j^{th} \) term: \( \omega^2 A_j = \frac{T}{ma} (-A_{j-1} + 2A_{j} - A_{j+1}) \)

\[
\omega^2 A_j = \frac{T}{ma} A_j (-e^{-ika} + 2 + e^{ika})
\]

\[
\omega^2 = \frac{T}{ma} \left( 2 - 2 \cos \frac{ka}{a} \right)
\]

\[
\omega_0 = \frac{T}{ma} = 2 \omega_0 \left( 1 - \cos \frac{ka}{a} \right)
\]

\[
\omega^2 = 4 \omega_0^2 \sin^2 \frac{ka}{2a}
\]
Almost the same as what we get in the last lecture!

\[ W = W(k) \] is a function of \( k \)

"Dispersion Relation"

\( k \) is given \( \Rightarrow \) \( W \) is determined

\( \Rightarrow \) wave number \( \Rightarrow \) angular frequency

\[ k = \frac{2\pi}{\lambda} \]

Normal modes: Standing waves!

Oscillating at frequency \( W \), determined by \( k \)

This system is infinitely long.

All possible \( k \) values (thus wavelength) are allowed. Each \( k \) value corresponds to a different normal mode with angular frequency \( W(k) \).
Now we will try to solve a finite system using this infinitely long system.

Consider the following boundary conditions:

1) Fixed end:

\[ a \quad a \quad \ldots \quad a \]

\[ y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_N \quad y_{N+1} \]

Boundary conditions: \( y_0 = 0 \), \( y_{N+1} = 0 \)

What are the normal modes satisfying the boundary condition?

There are two \( k \) values which give the same \( W \)

\( W(k) = W(-k) \)

Therefore: linear combinations of \( e^{ik} \) and \( e^{-ik} \) are also normal modes.

Guess: \( y_j = \text{Re} \left[ e^{i(Wt+\phi)} e^{ik} \right] \)

\( \alpha, \beta \) are constants.

Use boundary conditions:

\( y_0 = 0 \) \( \Rightarrow \alpha + \beta = 0 \) \( \Rightarrow \alpha = -\beta \)

\( y_{N+1} = 0 \) \( \Rightarrow \alpha \left( e^{i(N+1)ka} + e^{-i(N+1)ka} \right) = 0 \)

\( \Rightarrow \ k^2 a = \frac{\pi^2 n^2}{N+1} \) \( \Rightarrow \ k^2 a = \frac{n \pi^2}{N+1} \) \( n = 1, 2, 3, \ldots N \)
(More examples):

(2) Open End:

Boundary Conditions:
1. \( y_1 = y_0 \)
2. \( y_N = y_{N+1} \)

From first boundary condition:
\[
\begin{align*}
\alpha + \beta &= e^{ika} + e^{-ika} \\
&= \alpha (1 - e^{ika}) = \beta (e^{-ika} - 1)
\end{align*}
\]

Second boundary condition:
\[
\begin{align*}
\alpha e^{ika} + \beta e^{-ika} &= e^{ika} + e^{-ika} \\
\Rightarrow \alpha e^{ika} (1 - e^{-ika}) &= \beta e^{-ika} (1 - e^{ika}) \\
\Rightarrow e^{-ika} &= e^{ika} \\
\Rightarrow e^{2ika} &= 1 \\
\Rightarrow ka &= \frac{2n\pi}{\lambda N} = \frac{n\pi}{\lambda}
\end{align*}
\]

\( n = 1, 2, 3, 4, \ldots N \)

\[ \Rightarrow \beta = \alpha e^{ika} \Rightarrow y_j = \alpha (e^{ika} + e^{-i(j-1)ka}) \]

\[ = \alpha e^{-ika} (e^{i(j-1)ka} - e^{i(j-2)ka}) \]

\[ \propto \cos(ka(j-\frac{1}{2})) \]

(3) \[ y_0, y_N, \Delta \cos \omega t \]

Boundary Conditions:
1. \( y_0 = 0 \)
2. \( y_{N+1} = \Delta \cos \omega t \)

Need to find the "particular solution"

\( y_j \) must be oscillating at a frequency \( \omega t \).
What is the corresponding $R_d$ which gives $W_d$? Let $W(R)$

$$W_d^2 = 2W_0^2 \left(1 - \cos R_d a\right)$$

⇒ Solve to get $R_d a = \cos^{-1} \left(1 - \frac{W_d^2}{2W_0^2}\right)$

Guess $Y_j = \text{Re} \left[ e^{iW_0 t} \left( \alpha e^{iR_d a} + \beta e^{-iR_d a} \right) \right]$

Boundary condition at $j=0$

$$Y_0 = 0 \Rightarrow \alpha + \beta = 0 \Rightarrow \beta = -\alpha$$

⇒ $Y_j = \text{Re} \left[ 2i e^{iW_0 t} A \sin jR_d a \right]$ (Oscillatory at $W_d$)

Boundary condition at $j=N+1$

$$Y_{N+1} = Z \cos W_0 t = \text{Re} \left[ \Delta e^{iW_0 t} \right]$$

⇒ $2i A \sin (N+1) R_d a = \Delta$

$$A = \frac{\Delta}{2i \sin (N+1) R_d a}$$

⇒ $Y_j = \text{Re} \left[ \frac{\Delta \sin jR_d a}{\sin (N+1) R_d a} e^{iW_0 t} \right]$

$$= \frac{\Delta \sin jR_d a}{\sin (N+1) R_d a} \cos W_0 t$$

⇒ Explode when $R_d a = \frac{n\pi}{N+1}!!$ (Match with normal mode frequency)
Summary:

1. Symmetry + doesn't explode at the edge of the universe
   \[ \beta = \frac{iq}{a} \]

2. E.O.M can be derived from physical laws

3. Dispersion relation \( W(k) \) can be derived from (1) and (2)

4. The allowed \( k \) value is determined by boundary condition. Full solution = linear combination of normal modes

5. Use initial condition to determine the unknowns.
Now make it continuous!!

\[ M^{-1}KA \Rightarrow \omega^2 A_j = \frac{I}{m\alpha} (-A_{j-1} + 2A_j - A_{j+1}) \]

\[ M^{-1}KA \Rightarrow \omega^2 A(x) = -\frac{I}{m\alpha} (A(x-a) + 2A(x) - A(x+a)) \]

Taylor Series:

\[ f(x+\alpha x) = f(x) + \alpha x f'(x) + \frac{1}{2} \alpha^2 x^2 f''(x) \]

\[ A(x-a) = A(x) + a A'(x) + \frac{1}{2} a^2 A''(x) \]

\[ A(x+a) = A(x) + a A'(x) + \frac{1}{2} a^2 A''(x) \]

\[ \Rightarrow -A(x-a) + 2A(x) - A(x+a) = \frac{\partial^2 A(x)}{\partial x^2} a^2 + \ldots \]

\[ M^{-1}KA(x) = \frac{I}{m\alpha} \frac{\partial^2 A(x)}{\partial x^2} a^2 + \ldots \]

In the limit \( a \ll \text{wave length} \)

\[ \Rightarrow \text{we can ignore the higher order term} \]

\[ p_L = \frac{m}{\alpha} \]

\[ \Rightarrow M^{-1}K \rightarrow -\frac{T}{p_L} \frac{\partial^2}{\partial x^2} \]

\[ \Rightarrow \frac{\partial^2 u(x,t)}{\partial t^2} = t \frac{\partial^2 u(x,t)}{\partial x^2} \]

\[ \Rightarrow \text{Dispersion relation: } \omega = \sqrt{\frac{T}{p_L}} k^2 \]

\[ \frac{\omega}{k} = \nu_p = \sqrt{\frac{T}{p_L}} \]

\[ \nu_p: \text{phase velocity} \]

\[ \omega: \text{angular frequency} \]

\[ k: \text{wave number} \]