Exam 1

Last time:

System 1

\[
\begin{array}{cccccccc}
T & m & T & m & T & m & T & m \\
\hline
a & a & a & a & a & a & a & a \\
\end{array}
\]

\[
T \quad p_e = \frac{m}{a}
\]

\[
-a \rightarrow 0, \quad N \rightarrow \infty, \quad a \ll \frac{2\pi}{k} = \lambda
\]

\[
-\ddot{x} = M^TKx \\
M^TKA = \omega^2 A
\]

\[
\text{Jet term of } M^TKA: \quad \omega^2 A_j = \frac{T}{ma} (-A_{j+1} + 2A_j - A_{j-1})
\]

\[
\text{Continuum limit}
\]

\[
\omega^2 A(x) = \frac{T}{ma} \left(-A(x-a) + 2A(x) - A(x-a)\right)
\]

Taylor Series

\[
\approx \frac{T}{ma} \left(-\frac{\delta^2 A(x)}{\delta x^2} a^2\right)
\]

\[
= -\frac{T}{p_L} \frac{\delta^2 A(x)}{\delta x^2} \quad (2)
\]

\[
\Rightarrow M^TK \rightarrow -\frac{T}{p_L} \frac{\delta^2}{\delta x^2} \quad \psi_j \rightarrow \psi(x,t)
\]

From (1) and (2) \[
\frac{\delta^2 \psi(x,t)}{\delta t^2} = \frac{T}{p_L} \frac{\delta^2 \psi(x,t)}{\delta x^2}
\]
Original Dispersion Relation

\[ W^2 = 4 \frac{T}{ma} \sin^2 \frac{ka}{2} \]

\( a \ll \frac{2\pi}{k} \Rightarrow ka \text{ very small} \)

\[ \Rightarrow W^2 \approx 4 \frac{T}{ma} \left( \frac{ka}{2} \right)^2 = \frac{T}{f_L} k^2 \]

\[ \Rightarrow \ U_p = \frac{W}{k} = \sqrt{\frac{T}{f_L}} \]

\[ \Rightarrow \ \frac{\partial^2 \psi(x,t)}{\partial t^2} = U_p^2 \frac{\partial^2 \psi(x,t)}{\partial x^2} \quad \text{Wave Equation} \]

= Infinite number of coupled equations of motion.
Come back to the original question

What are the normal modes?

\[ \psi(x, t) = A(x) B(t) \]

\[ \uparrow \quad \text{control the time evolution} \]

Control the relative amplitude

Plug into wave equation

\[ A(x) \frac{\partial^2 B(t)}{\partial t^2} = \nu^2 B(t) \frac{\partial^2 A(x)}{\partial x^2} \]

\[ \frac{1}{\nu^2 B(t)} \frac{\partial^2 B(t)}{\partial t^2} = \frac{1}{A(x)} \frac{\partial^2 A(x)}{\partial x^2} \]

This Eq must be satisfied at all \( x \) and \( t \)!

\[ \Rightarrow \frac{1}{\nu^2 B(t)} \frac{\partial^2 B(t)}{\partial t^2} = \frac{1}{A(x)} \frac{\partial^2 A(x)}{\partial x^2} = -Km^2 \]

a constant
\[ 1 \frac{\dot{\ddot{B}(t)}}{V_p^2 B(t)} = -k_m^2 \]

\[ \ddot{B}(t) = -k_m^2 V_p^2 B(t) \]

\[ B(t) = B_m \sin(\omega_m t + \beta_m) \]

\[ \omega_m = V_p k_m \]

\[ 1 \frac{\dot{\ddot{A}(x)}}{A(x)} = -k_m^2 \]

\[ A(t) = C_m \sin(k_m x + \alpha_m) \]

\[ \psi_m(x,t) = A_m \sin(\omega_m t + \beta_m) \sin(k_m x + \alpha_m) \]

**Mth Normal mode.**

\[ \omega_m = V_p k_m \Rightarrow \text{decided by the property of the string} \]

The two unknowns: \( \alpha_m, k_m \), decided by the boundary conditions.

And \( A_m, \beta_m \), decided by the initial conditions.

\( \text{(will show that in the later discussion)} \)
Look at the structure of this normal mode solution. Let's stop and think about what we have learned:

1) Each point mass on the string is oscillating harmonically at the same frequency and phase! Only up and down, not in the horizontal direction!

2) Their relative amplitude = \sin function!

Need to determine unknown coefficients step by step.

Let's take a concrete example:

Suppose we have a string, one end fixed, the other end open.

Boundary condition:

1) At \( x=0 \) \( \Rightarrow \psi(0,t)=0 \)

2) At \( x=L \) \( \Rightarrow \frac{\partial \psi}{\partial x}(L,t)=0 \)

\( A \to \infty \), the ring is massless!

If \( \frac{\partial \psi}{\partial x} \neq 0 \) \( \Rightarrow \) Net force! (T and N do not cancel!)
What are the normal modes? Mode

\[(1) \Rightarrow \psi_m(0,t) = A_m \sin(\alpha_m) \sin(\omega_m t + \beta_m) = 0\]

\[\Rightarrow \alpha_m = 0\]

\[(2) \Rightarrow \frac{\partial \psi_m}{\partial x} = A_m k_m \sin(\omega_m t + \beta_m) \cos(k_m x + \alpha_m)\]

At \(x=L\): \(\frac{\partial \psi_m(L,t)}{\partial x} = 0 = A_m k_m \sin(\omega_m t + \beta_m) \cos(k_m L)\)

\[\Rightarrow k_m L = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{(2m-1)\pi}{2}\]

\[k_m = \frac{(2m-1)\pi}{2L}\]

\[m=1 \quad k_1 = \frac{\pi}{2L} \quad \lambda_1 = \frac{2\pi}{k_1} = 4L\]

\[\omega_1 = 2\pi k_1 = \sqrt{1\mu \frac{\pi}{2L}}\]
\[ m = 2 \quad k_2 = \frac{3\pi}{2L} \quad \lambda_2 = \frac{4}{3} L \]

\[ m = 3 \quad k_3 = \frac{5\pi}{2L} \quad \lambda_3 = \frac{4L}{5} \]

\[ \Rightarrow \text{General Solution:} \]

\[ \psi(x,t) = \sum_{m=1}^{\infty} A_m \sin(\omega_m t + \beta_m) \cdot \sin\left(\frac{k_m x}{L} + \gamma_m\right) \]

From boundary conditions:

\[ \alpha m = 0 \quad , \quad k_m = \frac{(2m-1)\pi}{2L} \]

\[ \psi(x,t) = \sum_{m=1}^{\infty} A_m \sin\left[\frac{(2m-1)\pi}{2L} \cdot t + \beta_m\right] \cdot \sin\left[\frac{(2m-1)\pi}{2L} \cdot x\right] \]
How do we extract $A_m$ and $\beta_m$?

$\sin(kx) \sin(y) = \frac{1}{2} \cos(xy) + \cos(xy)$

Suppose at $t=0$, the string looks like this. Also, the string is at rest ($u(x, 0) = 0$).

$\Rightarrow$ Initial conditions:

(a) $\psi(x, 0) = 0$  
(b) $\psi(x, 0)$ is known.

From (a) we get

$$
\psi(x, t) = \sum_{m=1}^{\infty} A_m \sin(k_m x + \beta_m) \cos(\omega_m t + \alpha_m)
$$

$\psi(x, 0) = 0 \Rightarrow \beta_m = \frac{\pi}{2} \Rightarrow \psi(x, 0) = \sum_{m=1}^{\infty} A_m \sin(k_m x + \alpha_m)$

**Demo**

(b) How do I extract $A_m$ from the given $\psi(x, 0)$?

$\Rightarrow$ Use the "orthogonality" of sine functions

$$
\int_0^L \sin(k_m x) \sin(k_n x) \, dx = \begin{cases} 
\frac{L}{2} & \text{if } m = n \\
0 & \text{if } m \neq n
\end{cases}
$$

$\Rightarrow$ We can extract $A_m$ by:

$$
A_m = \frac{2}{L} \int_0^L \psi(x, 0) \sin(k_m x) \, dx
$$

In this example:

$$
A_m = \frac{2}{L} \int_{\frac{1}{2}}^L h \sin(k_m x) \, dx
$$

$$
= \frac{2}{L} \left[ \frac{h}{k_m} \left[ \cos(k_m \frac{L}{2}) - \cos(k_m \frac{1}{2}) \right] \right]
$$

where $k_m = \frac{(2m-1)\pi}{2L}$