8.03 Lecture 12

Systems we have learned:

Wave equation:

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{v_p^2}{L^2} \frac{\partial^2 \psi}{\partial x^2}$$

There are three different kinds of systems discussed in the lecture:

1. String with constant tension and mass per unit length $\rho_L$
   $$v_p = \sqrt{\frac{T}{\rho_L}}$$

2. Spring with spring constant $k$, length $l$, and mass per unit length $\rho_L$
   $$v_p = \sqrt{\frac{kl}{\rho_L}}$$

3. Organ pipe with room pressure $P_0$ and air density $\rho$
   $$v_p = \sqrt{\frac{\gamma P_0}{\rho}}$$

This time, we are doing EM (electromagnetic) waves!
\[ \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \rightarrow \text{Gauss' Law} \]
\[ \nabla \cdot \vec{B} = 0 \rightarrow \text{Gauss' Law for magnetism} \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \text{Faraday's Law} \]
\[ \nabla \times \vec{B} = \mu_0 \left( \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \rightarrow \text{Ampere's Law} \]

In the vacuum: \( \rho = 0 \) and \( \vec{J} = 0 \) and we get:

\[ \nabla \cdot \vec{E} = 0 \]
\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
\[ \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \]

Where in the last two equations we see a changing magnetic field generates an electric field and a changing electric field generates a magnetic field. Can you see the EM wave solution from these equations? Maxwell saw it!

We need to use this identity:

\[ \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - (\nabla \cdot \nabla) \vec{A} \]

Where \( \nabla \cdot \nabla \equiv \nabla^2 \) is the Laplacian operator. In the vacuum:

\[ \nabla \times (\nabla \times \vec{E}) = \nabla (\nabla \cdot \vec{E}) - (\nabla^2) \vec{E} \]

Where we have made replacements based on the vacuum Maxwell equations above. Let's first examine the left hand side:

\[ \nabla \times -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} (\nabla \times \vec{B}) \]
\[ = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]
\[ = -\nabla^2 \vec{E} \]
\[ \Rightarrow \nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]
Recall
\[ \nabla^2 \equiv \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \]

And so we have a wave equation!!

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]

This equation changed the world! Maxwell is the first one who recognized it because of the term he put in. It was a wave equation with speed equal to the speed of light:

\[ v_p = c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 3 \cdot 10^8 \text{ m/s} \]

What about the \( \vec{B} \) field? We can do the same exercise:

\[ \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \]

It is very important that the associated magnetic field also satisfies the wave equation. From the Maxwell equation \( \vec{E} \) creates \( \vec{B} \) and \( \vec{B} \) creates \( \vec{E} \), therefore they can not exist without each other.

1638 Galileo: speed of light is large
1676 Romer: 2.2 \times 10^8 \text{ m/s}
1729 James Bradley: 3.01 \times 10^8 \text{ m/s}

This means that in vacuum you can excite EM waves! What is oscillating? The field!

Before we tackle EM waves, let’s review divergence and curl briefly.

*Field:
Scalar field: every position in the space gets a number. Temperature is an example.
Vector field: Instead of a number or scalar, every point gets a vector.

\[ \vec{A}(x, y, z) = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \]

The electric and magnetic fields are vector fields, e.g.:

\[ \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \]

To understand the structure of vector fields:
**Divergence** (using our definition of \( \nabla \) from above):

\[ \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \]
The divergence is a measure of how much the vector \( v \) spreads out \( (\text{diverges}) \) from a point:

The divergence of this vector field is positive.  The divergence of this vector field is zero.

\[ \nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z} \]

What exactly does curl mean? It is a measure of how much the vector \( \vec{A} \) “curls around” a point.

This vector field has a large curl.  This vector field has no curl.

\textbf{Gauss’ Theorem} (or the Divergence Theorem):

\[ \iiint_V (\nabla \cdot \vec{A}) \, d\tau = \iint_S \vec{A} \cdot \vec{d}a \]

Which allows us to relate the integral of the divergence over the whole volume (RHS) to a 2-D surface integral (LHS).

\textbf{Stokes’ Theorem:}

\[ \oint_S (\nabla \times \vec{A}) \cdot \vec{d}a = \oint_P \vec{A} \times \vec{dl} \]
Which allows us to relate the surface integral over the curl (LHS) to a line integral integral over a closed path (RHS).

*Consider a “plane wave” solution:

\[ \vec{E} = \text{Re} \left[ E_0 e^{i(kz - \omega t)} \hat{x} \right] \]

Only in the \( \hat{x} \) direction.

\[ = \{ E_0 \cos(kz - \omega t), 0, 0 \} \]

Check if it satisfies

\[ \nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]

\[ \Rightarrow \frac{\partial^2 E_x}{\partial z^2} \hat{x} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \hat{x} \]

In \( \hat{x} \) direction:

\[ -E_0 k^2 \cos(kz - \omega t) = -\mu_0 \varepsilon_0 \omega^2 E_0 \cos(kz - \omega t) \]

\[ \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c \Rightarrow \text{Condition needed to satisfy the wave equation.} \]

*What about \( \vec{B} \)?

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

\[ = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \frac{\partial E_x}{\partial z} \hat{y} - \frac{\partial E_y}{\partial z} \hat{z} \]

\[ = -k E_0 \sin(kz - \omega t) \hat{y} \]

\[ \Rightarrow \vec{B} = \frac{k}{\omega} E_0 \cos(kz - \omega t) \hat{y} = \frac{E_0}{c} \cos(kz - \omega t) \hat{y} \]

What did we learn from this exercise?

1. \( \vec{E} \) must come with \( \vec{B} \). In vacuum: the two fields are perpendicular and they are in phase.
   
   If \( \vec{k} \) is the direction of propagation then \( \vec{B} = \frac{1}{c} \vec{k} \times \vec{E} \) The amplitude of the magnetic field is equal to the amplitude of the electric field divided by the speed of light.

2. The EM wave is non-dispersive, meaning that the speed of the wave \( c \) is independent of the wave number \( k \): 

\[ \frac{\omega}{k} = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \]
3. The direction of the propagating EM wave is $\vec{E} \times \vec{B}$

In general a propagating EM wave can be written as:

$$\vec{E}(r, t) = \text{Re} \left[ \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t + \phi)} \right]$$

Where $\vec{E}_0 \equiv E_0 \hat{x} + E_0 \hat{y} + E_0 \hat{z}$, $\vec{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$ and $\omega \equiv ck$

Given this electric field, we can get the magnetic field:

$$\vec{B}(r, t) = \frac{1}{c} \vec{k} \times \vec{E}$$

Example:

$$\vec{k} = \frac{2\pi}{\lambda} \left\{ \frac{\hat{x}}{\sqrt{2}} + \frac{\hat{y}}{\sqrt{2}} \right\}$$

$$\vec{E}_0 = -\frac{E_0}{\sqrt{2}} \hat{x} + \frac{E_0}{\sqrt{2}} \hat{y}$$

$$\vec{k} \cdot \vec{r} = \frac{2\pi}{\sqrt{2}\lambda} (x + y)$$

$$\Rightarrow \vec{E}(x, y, z) = E_0 \left( -\frac{\hat{x}}{\sqrt{2}} + \frac{\hat{y}}{\sqrt{2}} \right) \cos \left( \frac{\sqrt{2}\pi}{\lambda} (x + y) - \omega t \right)$$
\[ \vec{B} = \frac{1}{c} \hat{k} \times \vec{E} \Rightarrow \vec{B}(x, y, z) = \frac{E_0}{c} \hat{z} \cos \left( \frac{\sqrt{2} \pi}{\lambda} (x + y) - \omega t \right) \]

If there is no other material, this EM wave will travel forever...

Now let’s put something into the game: A “perfect conductor”

A busy world inside this system! All the little charges are moving around without cost of energy (there is no dissipation).

Incident wave:

\[
\begin{cases}
\vec{E}_I = \frac{E_0}{2} \cos(kz - \omega t) \hat{x} \\
\vec{B}_I = \frac{E_0}{2c} \cos(kz - \omega t) \hat{y}
\end{cases}
\]

To satisfy the boundary conditions \( \vec{E} = 0 \) at \( z = 0 \) we need a reflected wave:

\[
\begin{align*}
\vec{E}_R &= -\frac{E_0}{2} \cos(-kz - \omega t) \hat{x} \\
\vec{B}_R &= \frac{E_0}{2c} \cos(-kz - \omega t) \hat{y}
\end{align*}
\]
\[\vec{E} = \vec{E}_I + \vec{E}_R = \frac{E_0}{2} (\cos(kz - \omega t) - \cos(-kz - \omega t)) \hat{x} = E_0 \sin(\omega t) \sin(kz) \hat{x}\]

\[\vec{B} = \vec{B}_I + \vec{B}_R = \frac{E_0}{2c} (\cos(kz - \omega t) + \cos(-kz - \omega t)) \hat{y} = \frac{E_0}{c} \cos(\omega t) \cos(kz) \hat{y}\]

Energy density?

\[U_E = \frac{1}{2} \varepsilon_0 E^2 = \frac{\varepsilon_0}{2} E_0^2 \sin^2 \omega t \sin^2 kz\]

\[U_B = \frac{1}{2 \mu_0} B^2 = \frac{\varepsilon_0}{2} E_0^2 \cos^2 \omega t \cos^2 kz\]

Poynting vector: directional energy flux, or the rate of energy transfer per unit area:

\[\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{1}{\mu_0} E_x B_y \hat{z} = \frac{E_0^2}{\mu_0 c} \sin \omega t \cos \omega t \sin k z \cos k z \hat{z} = \frac{E_0^2}{4 \mu_0 c} \sin(2 \omega t) \sin(2 k z) \hat{z}\]

This is how a microwave oven works!

*The EM waves are bounced around inside the oven
*EM waves increase the vibration of the molecules in the oven and increase the temperature of the food.