Reminder:

Maxwell's Equation in vacuum:

\[ \begin{align*}
\nabla \cdot E &= 0 \\
\nabla \cdot B &= 0 \\
\nabla \times E &= -\frac{\partial B}{\partial t} \\
\nabla \times B &= \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} = \frac{1}{C^2} \frac{\partial^2 E}{\partial t^2}
\end{align*} \]

\[ C = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \]

Resulting Wave Equations:

\[ \begin{align*}
\nabla^2 E &= \frac{1}{C^2} \frac{\partial^2 E}{\partial t^2} \\
\nabla^2 B &= \frac{1}{C^2} \frac{\partial^2 B}{\partial t^2}
\end{align*} \]

We discussed plane harmonic wave solution.

And you will show that the in general a progressing wave solution

\[ \vec{E} = E_0 \hat{y} f(x - vt) + \text{corresponding } \vec{B} \text{ field} \]

also satisfies Maxwell's equations.
How do we transmit "information"?

Simple harmonic wave: not useful.

We must use "pulses", chunks of localized energy in time.

For instance:

```
0 1 0 0 1 1 0
```

We have learned:

\[
\begin{align*}
  f(x - vt) & \text{ is a traveling wave moving} \\
  f(kx - \omega t) & \text{ in } +x \text{ direction and its shape is} \\
  & \text{kept unchanged if and only if we} \\
  & \text{are working on Non-dispersive medium.}
\end{align*}
\]

\[
\frac{\omega}{K} = v
\]
Consider an ideal string:

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \]

where \( \frac{w}{k} = c = \sqrt{\frac{T}{\rho_t}} \)

If we create a square pulse, the square pulse will move at a constant speed \( c \).

The shape of the square pulse doesn't change!

We call this string a non-dispersive medium and the "dispersion relation" is

\[ w = c \cdot k \]  \((\text{String tension is responsible for the restoring force})\)

However, if we consider the "stiffness" of the string,

( for example, piano string )

If we bend a piano string, even when there is no tension, the string wants to restore to its original shape.
Consider an ideal string: \( \frac{\omega}{K} = \frac{\nu}{K} = \sqrt{\frac{T}{\rho}} \)

\( \frac{\partial^2 \psi}{\partial t^2} = \nu^2 \frac{\partial^2 \psi}{\partial x^2} \)

\( \psi(x,t) \)

However, if we consider the "stiffness" of the string, make it more realistic.

\[
\frac{\partial^2 \psi}{\partial t^2} = \nu^2 \left[ \frac{\partial^4 \psi}{\partial x^4} - \alpha \frac{\partial^4 \psi}{\partial x^4} \right]
\]

(Dispersion relation becomes: \( (\text{Test function } A \cos(kx-\omega t)) \))

\[
\omega^2 = \nu^2 (k^2 + \alpha k^4)
\]

\( \Rightarrow \frac{\omega}{K} = \nu \sqrt{1 + \alpha k^2} \quad \text{Not a constant} \quad \text{v.s } k !! \)

\( K \nu \sqrt{1 + \alpha k^2} \)

\( K \nu \)

\( K = \frac{2\pi}{\lambda} \)

Large \( k \Rightarrow \) short \( \lambda \Rightarrow \) a lot of distortion

\( \Rightarrow \) higher speed \( \nu \)
Consequence:

Components with different $k$ will be moving at different speed $V_p = \frac{WK_0}{k}$ if we consider $S \sin (kx - Wc_k t)$

$\Rightarrow$ Dispersion!

(Demo)

<table>
<thead>
<tr>
<th>Dispersion</th>
<th>Mode</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>0.02</td>
</tr>
</tbody>
</table>
Dispersion: a variation of wave speed with wave length.

Example: Addition of two progressing waves:

\[ \psi_1(x,t) = A \sin (k_1x - \omega_1 t) \quad U_1 = \frac{\omega_1}{k_1} \]

\[ \psi_2(x,t) = A \sin (k_2x - \omega_2 t) \quad U_2 = \frac{\omega_2}{k_2} \]

Adding \( \psi_1 \) and \( \psi_2 \):

\[ \sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) \]

\[ \Rightarrow \psi = \psi_1 + \psi_2 \]

\[ = 2A \sin \left( \frac{k_1+k_2}{2} x - \frac{\omega_1+\omega_2}{2} t \right) \cos \left( \frac{k_1-k_2}{2} x - \frac{\omega_1-\omega_2}{2} t \right) \]

Assuming \( k_1 \approx k_2 \approx k \quad \omega_1 \approx \omega_2 \approx \omega \)

Amplitude Modulation

Phase velocity: \( v_p = \frac{\omega}{k} \)

Group velocity: \( v_g = \frac{(\omega_1 - \omega_2)}{(k_1 - k_2)} \approx \frac{d\omega}{dk} \)
Bounded system: $S, T, \alpha$

\[ U(x,t) = \sum_{m} A_m \sin(K_m x + \alpha_m) \sin(W_m t + \beta_m) \]

\[ W_m = W(K_m) \]

Then evolve as a function of time!
Now consider the boundary condition of this system:

\[ \psi(0, t) = 0 \quad \text{and} \quad \psi(L, t) = 0 \]

Similar to what we have solved before:

\[ \lambda_m = \frac{m \pi}{L}, \quad \alpha_m = 0 \]

Identical to the ideal spring case (\( \alpha = 0 \))

We learned that:

1. The boundary condition "set" the \( \lambda_m \)!
   - Does not depend on the dispersion relation \( \omega(\lambda) \)

2. The individual normal modes are oscillating
   - at \( \omega_m = \omega(\lambda_m) \) calculated by the dispersion relation:
   - Does depend on the dispersion relation!
If we plot the dispersion relation:

\[ W_m \]

\[ K, K_a, K_b, K_c \]

equally spaced! But \( W_m \) is not equally spaced.

**Full solution**

\[ \Psi(x,t) = \sum_{m} A_m \sin(K_m x + \alpha_m) \sin(W_m t + \beta_m) \]

\[ = \sum_{m} \Psi_m \]

**Ex:** \( \Psi(x,t) = \Psi_1 + \Psi_2 \)

In non-dispersive medium: the system go back to the original shape after \( \frac{2\pi}{W_i} \)

In dispersive medium: \( W_2 \neq W \),

Need to wait longer until the \( T = \text{least common multiple of} \ \frac{2\pi}{W_1} \text{ and } \frac{2\pi}{W_2} \)