In many media: \( v = v(\lambda) \)

For instance: light in glass

* Speed of wave propagation depends on wavelength \( \lambda \) (or \( \omega \) or \( k \))
  - Red > Violet

* Deep water * Beaded string * realistic string *

\[ \text{Non-dispersive medium} \]

\[ \text{Dispersive medium} \]

made of many different modes \( \rightarrow \) traveling at different speed \( \rightarrow \) "disperse"

Phase velocity: \( v_p = \frac{\omega}{k} \)

Group velocity: \( v_g = \frac{d\omega}{dk} \)

* Non-dispersive medium \( \omega = v \cdot k \)

\( \Rightarrow \) Phase velocity = \( v \)

Group velocity \( \frac{d\omega}{dk} = v = \text{Phase velocity} \)
Example:

EM wave passing through an ionic crystal

The dispersion curve looks like

\[ \begin{align*}
  \omega & \quad \text{versus} \quad k \\
\end{align*} \]

(1) What is the group velocity and phase velocity as a function of \( k \)?

\[ \begin{align*}
  V & \quad \text{versus} \quad k \\
U & \quad \text{versus} \quad k \\
\end{align*} \]

(2) Velocity \( v \) versus \( \omega \)?

\[ \begin{align*}
  V & \quad \text{versus} \quad \omega \\
U & \quad \text{versus} \quad \omega \\
\end{align*} \]

(3) What will happen to radiation striking such a crystal if the frequency is \( \omega < \omega_b \)?

There is no propagation or loss in this crystal.

\[ \Rightarrow \text{ totally reflected!} \]
If we have a very long string:

\[ f(t) \]

We shake one end. \( \Rightarrow \) Produce a progressive wave!

\[ u(x, t) = f(t - \frac{x}{v}) \]

for non-dispersive medium.

How about dispersive medium?

\( \Rightarrow \) Waves with different frequency (or wave length) are traveling at different speed!

\( \Rightarrow \) Need to decompose \( f(t) \) into waves with fixed frequency

Then attack them one by one!!!

💡 Use Fourier transform this mathematical tool!

\[ f(t) = \int_{-\infty}^{\infty} dw \ C(w) \ e^{-i\omega t} \]

Amplitude oscillation at \( \omega \)
After we decompose \( f(t) \) into many harmonic oscillations with different frequency and use the dispersion relation \( \omega = \omega(k) \)

\[
\Psi(x,t) = \int_{-\infty}^{\infty} dw \ C(w) \ e^{-i\omega t + ikw x}
\]

With a given \( \omega \) we can solve \( \Psi \) (a function of \( \omega \)).

**Special Case: Non-dispersive system:**

\[
\Psi(x,t) = \int_{-\infty}^{\infty} dw \ C(w) \ e^{-i\omega t - \frac{\omega}{v} x}
\]

\[
= \int_{-\infty}^{\infty} dw \ C(w) \ e^{-i\omega (t - \frac{x}{v})}
\]

\[
= f(t - \frac{x}{v})
\]

Make sense!
How do we determine \( C(w) \)?

Orthogonality:

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(w-w')t} dt = \delta(w-w')
\]

\( \delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \)

Some useful formula:

\[
\int_{-\infty}^{\infty} \delta(x) dx = 1
\]

\[
\int_{-\infty}^{\infty} \delta(x-x') f(x') dx' = f(x)
\]

Now if I calculate this quantity:

\[
\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \ f(t) e^{i(wt)}
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \left( \int_{-\infty}^{\infty} C(w) e^{-iwt} \right) e^{iwt}
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} C(w') dw' \int_{-\infty}^{\infty} dt e^{i(w-w')t}
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} C(w') \delta(w-w') dw' = C(w)
\]
Now we have a problem:

\[ f_s(t) \]

How do we overcome this difficulty?

A smart idea: AM radio

\[ f_s(t) : \text{the signal we want to transmit.} \]

For instance: music, sound \( \sim 1 \text{ kHz} \)

Carrier: \( \cos \omega t \) or \( e^{i\omega t} \)

The frequency of the AM radio \( \sim 0.1 \text{ to } 30 \text{ MHz} \)

Instead of transmitting \( f_s(t) \) directly (we know this doesn't work), we transmit

\[ f(t) = f_s(t) \cos \omega t \]

Since \( f_s(t) \) is SLOW compared to \( \cos \omega t \) (\( \sim 1 \text{ kHz} \))

\( \cos \omega t \sim 0.1 \text{ to } 30 \text{ MHz} \)

\( \Rightarrow \) the resulting \( W \) range with non-zero \( C(W) \) is "narrow"
This is because:

\[
\cos W_s t \cos W_0 t = \frac{1}{2} \left[ \cos (W_0 + W_s) t + \cos (W_0 - W_s) t \right]
\]

\(W_s\) is the "typical frequency" of the signal.
\(W_0\) is the carrier frequency.

Therefore, the range of \(W\) with non-zero \(C(W)\) is \(\sim W_0 - W_s\) to \(W_0 + W_s\).

Where \(W_s\) is \(\ll W_0\).

Dispersion Relation: \(W = W(k)\)
Suppose $W(k)$ is slowly varying around $W_0$

$$W = W(k) = W_0 + (k-k_0) \frac{\partial W}{\partial k} \bigg|_{k=k_0} + ...$$

$$\Rightarrow W \approx W_0 + (k-k_0) V_g$$

$W_0 = W(k_0)$ when $k \approx k_0$, $W \approx W_0$.

$\Rightarrow$ Higher order terms are negligible in the range $W_0 - \Delta W < W < W_0 + \Delta W$.

If my $f(t)$ satisfies with $C(W) \approx 0$ for $|W-W_0| > \Delta W$.

ie. we are looking at $f(t)$ with the corresponding $C(W)$ like:

$|C(W)|$ on a log scale, nonzero only inside this window.
\( f_s(t) \) must be "slowly-varying" compared to the carrier wave \( e^{-i\omega t} \)

\[ f(t) = e(f_s(t) e^{-i\omega t}) \]

"envelope"  "carrier"

This is actually "Amplitude Modulation"

AM radio!

If \( \Delta W \ll \omega_0 \) (a small window with \( C(w) \to 0 \))

\[ \Rightarrow \text{higher order terms in } W(K) \text{ doesn't matter or negligible} \]

from P10 (a)

\[ W = \nu_g K + A \quad \quad A = \omega_0 - 2\nu K_0 \]

\[ K = \frac{W}{\nu_g} + b \quad \quad b = K_0 - \omega_0/\nu_g \]

\( A \) and \( b \) are constants.

Now we want to show

\[ \Psi(x,t) = e^{i(f_s(t - \frac{x}{\nu_g}) e^{-i(\omega_0 t - K_0 x)}} \]
Fourier transform: we can rewrite $f_s(t)$ as

$$f_s(t) = \int_{-\infty}^{\infty} dw \ c(w) \ e^{-iwt}$$

Make AM radio multiply by $e^{-iwt}$

$$f(t) = \int_{-\infty}^{\infty} dw \ c(w) \ e^{-i(wt+wo)t}$$

Let $f_s(t) e^{-iwt}$

$$= \int_{-\infty}^{\infty} dw \ c(w-w_0) e^{-iwt}$$

Propagate to all $x$

$$\mathcal{U}(x,t) = \text{Re} \left[ \int_{-\infty}^{\infty} dw \ c(w-w_0) e^{-iwt} e^{ikx} \right]$$

C($w$) is nonzero around $w_0$

(b) $\approx \int_{-\infty}^{\infty} dw \ c(w-w_0) e^{-iwt} e^{i(w/2y+b)x}$

(b) $\approx \int_{-\infty}^{\infty} dw \ c(w-w_0) e^{-i(w(t-\frac{x}{2y})+ibx}$

(b) $\approx \int_{-\infty}^{\infty} dw \ c(w) e^{-i(wt+wo)(t-\frac{x}{2y})+ibx}$

(b) $\approx \int_{-\infty}^{\infty} dw \ c(w) e^{-i(w(t-\frac{x}{2y})+ibx}$

Therefore

$$\mathcal{U}(x,t) = \text{Re} \left[ f_s(t-\frac{x}{2y}) e^{-i(wo t-k_0 x)} \right]$$
\[ \Psi(x,t) = \text{Re}\left[ f_s(t - \frac{x}{V_g}) \ e^{-i(\omega t - k_0 x)} \right] \]

The envelope traveling at \( V_g \)

The Carrier traveling at \( V_p \)

Group velocity!

Phase Velocity!

What is the typical carrier frequency?

Medium Frequency 300kHz \( \leftrightarrow \) 3MHz \( \rightarrow \) Skywave

High Frequency 3MHz \( \leftrightarrow \) 30MHz

The envelope shape doesn't change!!

(No dispersion)

Enable us to send voice, music to places which are thousands of miles away!!