8.03 Lecture 16

We have discussed this system in lecture 8:

Mass can only move up and down in the \( \hat{y} \) direction. We have solved it by “space translation symmetry.” We obtained the dispersion relation:

\[
\omega(k) = \frac{T}{ma} \sin \left( \frac{ka}{2} \right)
\]

Where \( T \) is string tension, \( m \) is mass, \( a \) is the distance between masses at equilibrium. Eigenvectors (where \( j \) is the label of the mass):

\[
e^{ikj\cdot a}
\]

Today we are doing 2D and 3D system!! In general, we don’t know how to solve those systems! :( But we know how to solve highly symmetric systems!! If we consider an infinitely long array of masses:

Where \( m \) is the mass, \( T_V, T_H \) are the tensions, and we have ideal strings. Particles can only move in the \( \hat{z} \) direction. Good news: space translation symmetry! Eigenvectors:

\[
e^{ik_x x} e^{ik_y y}
\]

Where \( x \equiv j_x a_H \) and \( y = j_y a_V \) and \( (j_x, j_y) \) are indices.

\[
\Rightarrow \psi(x, y) = Ae^{ik_x x} e^{ik_y y} = Ae^{i\vec{k} \cdot \vec{r}}
\]
We can use the expression above to get the dispersion relation:

\[ \omega^2 = \frac{4T_H}{ma_H} \sin^2 \left( \frac{k_x a_H}{2} \right) + \frac{4T_V}{ma_V} \sin^2 \left( \frac{k_y a_V}{2} \right) \]

This is a dispersive medium because \( \frac{\omega}{|k|} \) is not a constant.

At fixed \( \omega \): If we consider a 1D bead-string system:

\[ e^{ikx} \text{ and } e^{-ikx} \]

This is \( \cos(kx) \) and \( \sin(kx) \)!!

\[
\cos(kx) = \frac{1}{2}(e^{ikx} + e^{-ikx}) \\
\sin(kx) = \frac{1}{2i}(e^{ikx} - e^{-ikx})
\]

We know from the discussion above, the eigenvector of \( M^{-1}k \) matrix is sin or cos. Back to the two-dimensional case: If we fix the angular frequency to be \( \omega \). There are multiple values of \( k_x \) and \( k_y \) which can give the same \( \omega \) (actually infinite number of choices). This is because \( k_x \) and \( k_y \) are continuous: can be any value before we introduce boundary conditions. If we lower \( k_x \) a bit we can increase \( k_y \) to compensate! Example: if I have dispersion relation of this form:

\[ \omega^2 = 5 \sin^2 k_x + 5 \sin^2 k_y \]

There are many possible pairs of \( k_x \) and \( k_y \) which gives the same \( \omega \)!!

Now we add the wall back in:

\[ \psi(0, y, t) = \psi(L_H, y, t) = \psi(x, 0, t) = \psi(x, L_V, t) = 0 \]

In this example: \( L_H = 5a_H \) and \( L_V = 4a_V \)
There are now only four modes of the finite system with the same $\omega$

$$A e^{\pm ik_x x} e^{\pm ik_y y}$$

$$k_x = \frac{n_x \pi}{L_H} \quad k_y = \frac{n_y \pi}{L_V}$$

$$L_H = 5a_H \quad L_V = 4a_V$$

and $n_x$ runs from 1 to 4 while $n_y$ runs from 1 to 3. Linear combinations of

$$e^{+ik_x x} e^{+ik_y y}, \quad e^{+ik_x x} e^{-ik_y y}, \quad e^{-ik_x x} e^{-ik_y y}$$

gives $A \sin k_x x \sin k_y y$ which satisfy the boundary conditions.

$$\Rightarrow \psi(n_x, n_y)(x, y, t) = A(n_x, n_y) \sin \left( \frac{n_x \pi x}{L_H} \right) \sin \left( \frac{n_y \pi y}{L_V} \right)$$

Discrete case general solution:

$$\psi(x, y, t) = \sum_{n_x, n_y} A(n_x, n_y) \sin \left( \frac{n_x \pi x}{L_H} \right) \sin \left( \frac{n_y \pi y}{L_V} \right)$$

Continuous case (assuming $T_H = T_V = T$) $a_H = a_V \rightarrow 0$

$$\omega^2 = \frac{4T k_x^2 a^2}{ma} + \frac{4T k_y^2 a^2}{ma}$$

$$= \frac{T a}{m} (k_x^2 + k_y^2)$$

Define the surface mass density, $\rho = m/a^2$, and the surface tension, $T_s = T/a$

$$\omega^2 = \frac{T_s}{\rho_s} (k_x^2 + k_y^2) = \frac{T_s}{\rho_s} |\vec{k}|^2$$

Similar to one dimensional case. Continuous limit gives:

$$\frac{\partial^2}{\partial t^2} \psi(x, y, t) = v^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(x, y, t)$$

$$= v^2 \nabla^2 \psi(x, y, t)$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} \psi(x, y, t) = v^2 \nabla^2 \psi(x, y, t)$$

$$\psi \propto A \sin(k_x x) \sin(k_y y) \sin(\omega t + \phi)$$
Where $v = \sqrt{T_s/rho_s}$. Similarly in the 3D case:

$$\frac{\partial^2}{\partial t^2} \psi(x, y, z, t) = v^2 \nabla^2 \psi(x, y, z, t)$$

Continuous case: 3D sound wave. Example: sound wave in a box

Example:

$$k_x = \frac{n_x \pi}{a} \quad k_y = \frac{n_y \pi}{b} \quad k_z = \frac{n_z \pi}{c}$$

Guess

$$\vec{\psi} \propto \sin(k_xx) \sin(k_yy) \sin(k_zz) \sin(\omega t + \phi)$$

Plug into wave equation:

$$\omega^2 = v^2 (k_x^2 + k_y^2 + k_z^2)$$

$$= v^2 \left( \left( \frac{n_x \pi}{a} \right)^2 + \left( \frac{n_y \pi}{b} \right)^2 + \left( \frac{n_z \pi}{c} \right)^2 \right)$$

Where $n_x, n_y, \text{ and } n_z$ are integers.

2 and 3D progressive wave:
Simple example: “plane waves”

$$\psi(\vec{r}, t) = Ae^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$\vec{k}$: direction of propagation

"Wave number vector"

$$\lambda = \frac{2\pi}{|\vec{k}|}$$
This can be used to describe EM waves, sound waves, or waves on membranes. If there is no other medium, this wave will continue forever. Let's continue a 2D membrane stretched in the $z = 0$ plane with surface mass density $\rho_s$ and surface tension $T_s$

$$\omega^2 = v^2(k_x^2 + k_y^2)$$

and waves will travel at speed $v = \sqrt{\frac{T_s}{\rho_s}}$. To add some excitement:

We place a second membrane on the other side, and our wave approaches this membrane. What will happen? One would usually expect an incident wave to produce a reflected and transmitted wave.

$$\psi_L = A e^{i(\vec{k} \cdot \vec{r} - \omega t)} + \sum_{\alpha} R_{\alpha} A e^{i(\vec{k}_\alpha \cdot \vec{r} - \omega t)}$$ \hspace{1cm} (x \leq 0)$$

$$\psi_R = \sum_{\beta} T_{\beta} A e^{i(\vec{k}_\beta \cdot \vec{r} - \omega t)}$$ \hspace{1cm} (x \geq 0)$$
Where $\sum_\alpha$ and $\sum_\beta$ sum over all possible $\vec{k}_\alpha$ and $\vec{k}_\beta$ which give angular frequency $\omega$

$$|k_\alpha|^2 = \omega^2 \frac{\rho_s}{T_s} = \frac{\omega^2}{v^2}, \quad |k_\beta|^2 = \omega^2 \frac{\rho'_s}{T'_s} = \frac{\omega^2}{v'^2}$$

To calculate $R_\alpha$ and $T_\beta$ as well as $\vec{k}_\alpha$ and $\vec{k}_\beta$ we need boundary conditions!
At $x = z = 0$ the membrane cannot break so we need $\psi_L = \psi_R$

$$\psi(0, y, 0, t) = Ae^{i(k_yy - \omega t)} + \sum_\alpha R_\alpha Ae^{i(k_{\alpha y}y - \omega t)} = \sum_\beta T_\beta Ae^{i(k_{\beta y}y - \omega t)}$$

Where the equality is established with the boundary condition. This can only be true when $k_{\alpha y} = k_{\beta y} = k_y$. Only when

$$k_{\alpha x} = -\sqrt{\omega^2/v^2 - k_y^2} = -k_x \quad \text{and} \quad k_{\beta x} = \sqrt{\omega^2/v'^2 - k_y^2}$$

We can satisfy the boundary condition.
We have $|\vec{k}| \sin \theta = |\vec{k}'| \sin \theta'$

\[
\begin{align*}
n &= \frac{c}{v} = \frac{c}{\omega} |k| \\
n' &= \frac{c}{v'} = \frac{c}{\omega} |k'| \\
\Rightarrow n \sin \theta &= n' \sin \theta'
\end{align*}
\]

Snell’s Law! We have just proved the two MOST IMPORTANT LAWS of geometrical optics!!!

(1.) Reflection: $\theta_1 = \theta_2$

(2.) Snell’s Law: $n_1 \sin \theta_1 = n_2 \sin \theta_2$ where $n$ is a refraction index

(3.) It works for water, glass, sound, and light waves!

(4.) If we continue to increase $\theta_1$ then \[ \frac{n_1}{n_2} \sin \theta_1 > 1 \]

There is no transmission!