8.03 Lecture 18

Waveplate: use material which the index of reflection is different for different orientations of light passing through it!

\[ \Delta \phi = \frac{2\pi l}{\lambda_x} - \frac{2\pi l}{\lambda_y} = \frac{n_x - n_y}{c} \omega l \]

Quarter-waveplate: \( \Delta \phi \) is designed to be \( \frac{\pi}{2} \)

*Axis with smaller phase \( \rightarrow \) fast axis

*Axis with larger phase \( \rightarrow \) slow axis

Circularly polarized
Matrix: \( Q_0 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \)

In general:
\[
\begin{pmatrix}
\cos^2 \theta + i \sin^2 \theta & \cos \theta \sin \theta - i \sin \theta \cos \theta \\
\cos \theta \sin \theta - i \sin \theta \cos \theta & \sin^2 \theta + i \cos^2 \theta 
\end{pmatrix}
\]

Where \( \theta \) is the direction of the fast axis with respect to the \( x \) axis.
(Editor’s note: see video lecture for a demonstration.)

How do we produce EM waves? Radiation from a point source.
In vacuum, EM wave neither loses nor gains energy. Recall the Poynting vector: \( \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \)
“rate of energy transfer per area”

\[
\langle S \cdot A \rangle_1 = \langle S \cdot A \rangle_2 = \text{power}
\]
\[
\langle S \rangle \propto \frac{1}{A} \propto \frac{1}{r^2}
\]
\[
\Rightarrow \langle \vec{E} \rangle, \langle \vec{B} \rangle \propto \frac{1}{r}
\]

Question: How do I produce radiation?

i Stationary charge:

\[
\vec{E} = \frac{q}{4 \pi \epsilon_0 r^2} \propto \frac{1}{r^2}
\]
\[
\vec{B}_0 = 0
\]

\( \vec{S} = 0 \)
ii Charge at constant speed $u$:

$$\beta = \frac{u}{c}$$

$$\vec{E} = \frac{q}{4\pi \epsilon_0 r^2} \frac{1 - \beta^2}{\left(1 - \beta^2 \sin^2 \theta\right)^{3/2}} \hat{r}$$

$$\vec{B} = \frac{\vec{u} \times \vec{E}}{c^2} \propto \frac{1}{r^2}$$

$$\Rightarrow |\vec{E}| \propto \frac{1}{r^2}, \quad |\vec{B}| \propto \frac{1}{r^2}$$

$$\frac{1}{\mu_0} \vec{E} \times \vec{B} = \vec{S} \propto \frac{1}{r^4} \Rightarrow \text{Does not radiate}$$

(Or we can use a simpler argument: boost to the rest frame of the charge)

Therefore we need to accelerate the charge to produce radiation. (Proof can be found in Georgi 355-360). Or the following geometrical argument. Goal: to create a “kink” in the electric field: Accelerated Charge!

Consider a charge, accelerated between $t = 0$ to $t = \Delta t$. $a$ is small and $\Delta t$ is small.

(1) Surface: information that the charge accelerated has only just reached this sphere

(2) Surface: information that the charge moving with constant velocity has reached this sphere

Q: What will the “observer” see at $t = t + \Delta t$? A: A stationary charge.

Therefore outside (1) the electric field is like the charge has never moved (where the observer lives). Inside (2) the electric field is in the $\hat{r}$ direction. Between (1) and (2) the field must be continuous because there is no source between them. Since $u \equiv a \cdot \Delta t$ is $\ll c$ (where $u$ is the velocity of the
charge after acceleration) then the field lines from A to B are approximately parallel. We have managed to create a “kink”!

\[ U = a \cdot \Delta t \]

\[ \frac{E_\perp}{E_\parallel} = -\frac{U_\Delta t}{\cot} = \frac{-a \Delta t \cdot t}{\cot} \]

\[ \Rightarrow E_\perp = \frac{-a_\perp r}{c^2} E_\parallel \]

What is \( E_\parallel \)? Use Gauss’ Law:

\[ E_\parallel = E_{\text{Out}} = \frac{q}{4\pi\epsilon_0 r^2} = \text{Electric field outside} \]

\[ E_\perp = \frac{-qa_\perp}{4\pi\epsilon_0 r^2 c^2} \]
This is very important! $E_\perp$ at position $\vec{r}$ is due to acceleration which occurred at a retarded time:

$$ t' = t - r/c $$

$$ \Rightarrow \vec{E}_{\text{Rad}}(\vec{r},t) = \frac{-q\vec{a}_\perp(t - r/c)}{4\pi\varepsilon_0 c^2 r} $$

$$ \Rightarrow \vec{B}_{\text{Rad}} \propto \frac{1}{r} $$

$$ \Rightarrow \vec{S}_{\text{Rad}} \propto \vec{E}_{\text{Rad}} \times \vec{B}_{\text{Rad}} \propto \frac{1}{r^2} $$

We are sending energy to the edge of the universe!!

$$ \vec{r} \gg \text{scale of } \vec{a}(t) \text{ such that the static contributions die out.} $$

$$ \vec{E}_{\text{Rad}}(\vec{r},t) = \frac{-q\vec{a}_\perp(t - r/c)}{4\pi\varepsilon_0 c^2 r} $$

$$ \vec{B}_{\text{Rad}}(\vec{r},t) = \frac{1}{c} \hat{r} \times \vec{E}_{\text{Rad}}(\vec{r},t) $$

$$ \vec{S}_{\text{Rad}}(\vec{r},t) = \frac{1}{\mu_0} \vec{E}_{\text{Rad}} \times \vec{B}_{\text{Rad}} $$

$$ \vec{a}_\perp = \vec{a} - \vec{a} \cdot \hat{r} \hat{r}, \quad \hat{r} = \frac{\vec{r}}{|\vec{r}|} $$

1. Get $\vec{a}$
2. define $\vec{r}$, get $\vec{a}_\perp \quad \vec{a}_\perp = \vec{a} - \vec{a} \cdot \hat{r} \hat{r}$
3. $\vec{E}_{\text{Rad}}$
4. $\vec{B}_{\text{Rad}} = \frac{1}{c} \hat{r} \times \vec{E}_{\text{Rad}}$
5. $\vec{S}_{\text{Rad}} = \frac{1}{\mu_0} \vec{E}_{\text{Rad}} \times \vec{B}_{\text{Rad}}$

6. Total power: $P(t) = \iiint \vec{S}_{\text{Rad}}(\vec{r},t) \cdot dA \hat{n} = \frac{q^2 |a(t - r/c)|^2}{4\pi\varepsilon_0 c^3}$
Example: harmonically oscillating charge:

\[ x = \hat{z}d \cos \omega t \] and \( R \gg d \)

1. At a distance \( R \) away from the charge in the \( \hat{z} \):

\[
\ddot{a}(t) = \ddot{x}(t) = -\hat{z}d\omega^2 \cos \omega t
\]

\[
\vec{E}_{Rad}(\vec{r}, t) = -\frac{q\vec{a}_\perp(t - r/c)}{4\pi\epsilon_0 c^2 r}
\]

\[
\vec{a}_\perp = \vec{a} - \vec{a} \cdot \hat{r} \hat{r} \quad \text{in this case } \vec{a} \parallel \hat{z}
\]

\[ \Rightarrow \vec{a}_\perp = 0 \]

\[ \Rightarrow \text{No radiation!} \]

2. How about \( R\hat{y} \)?

\[
\vec{a}_\perp = \vec{a} - \vec{a} \cdot \hat{y} \hat{y} = -\hat{z}d\omega^2 \cos \omega t
\]

\[
\vec{E}_{Rad}(t) = +\frac{qd\omega^2 \cos(\omega(t - R/c))}{4\pi\epsilon_0 c^2 R} \hat{z}
\]

\[
\vec{B}_{Rad}(t) = \frac{1}{c} \hat{y} \times \vec{E}_{Rad}(t) = \frac{qd\omega^2 \cos(\omega(t - R/c))}{4\pi\epsilon_0 c^3 R} \hat{x}
\]

We get harmonic waves with amplitude decreasing versus \( R \)

3. How about at \( R \left( \frac{1}{2} \hat{y} + \frac{\sqrt{3}}{2} \hat{z} \right) \)?
(30° angle with respect to the z-axis in the y − z plane)

\[ \vec{a}_\perp(t) = \vec{a} - (\vec{a} \cdot \hat{r})\hat{r} \]
\[ = -\omega^2 d \cos(\omega t) \left( \frac{\hat{z}}{2} - \frac{\sqrt{3}}{2} \left( \frac{1}{2}\hat{y} + \frac{\sqrt{3}}{2}\hat{z} \right) \right) \]
\[ = -\omega^2 d \cos(\omega t) \left( \frac{\hat{z}}{4} - \frac{\sqrt{3}}{4}\hat{y} \right) \]

\[ \vec{E}_{\text{Rad}} = \frac{q\omega^2 d}{8\pi\varepsilon_0 c^3 R} \cos(\omega(t - R/c)) \left( \frac{1}{2}\hat{z} - \frac{\sqrt{3}}{2}\hat{y} \right) \]

Example 2: A particle with charge \( q \) is moving on an elliptical orbit

\[
\begin{align*}
  x(t) &= \sqrt{2}A \cos(\omega t) \\
  y(t) &= A \sin(\omega t)
\end{align*}
\]

What are the polarizations of the electric field seen by distant observers on the positive \( x, y, z \) axes? First calculate \( \vec{a}(t) \)

\[ \vec{a}(t) = -\sqrt{2}A\omega^2 \cos(\omega t)\hat{x} - A\omega^2 \sin(\omega t)\hat{y} \]

(1) Observer \( R\hat{x} \)

\[ \vec{a}_\perp = -A\omega^2 \sin(\omega t)\hat{y} \]
\[ \vec{E}_{\text{Rad}} = \frac{q\omega^2 A}{4\pi\varepsilon_0 c^3 R} \sin(\omega(t - R/c)) \text{ Linearly polarized} \]

(2) \( \hat{y} \): similarly, also linearly polarized
(3) \( \hat{z} \): elliptically polarized