Waveplate: use materials which the index of reflection is different for different orientations of light passing it!

\[ k_x = \frac{n_x}{c} \omega = \frac{2\pi}{\lambda_x} \]
\[ k_y = \frac{n_y}{c} \omega = \frac{2\pi}{\lambda_y} \]

\[ \Delta \phi = \frac{2\pi l}{\lambda_x} - \frac{2\pi l}{\lambda_y} = \left( \frac{n_x - n_y}{c} \right) \omega l \]

Quarter-wave plate: \( \Delta \phi \) is designed to be \( \frac{\pi}{2} \)

Axis with smaller phase
\( \rightarrow \) fast axis
Axis with larger phase
\( \rightarrow \) slow axis

Circularly polarized
Matrix: \[ Q_0 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \]

In general:
\[
Q_0 = \begin{pmatrix} \cos^2 \Theta + i \sin^2 \Theta & \cos \Theta \sin \Theta - i \sin \Theta \cos \Theta \\ \cos \Theta \sin \Theta - i \sin \Theta \cos \Theta & \sin^2 \Theta + i \cos^2 \Theta \end{pmatrix}
\]

\( \Theta \): the direction of fast axis with respect to \( x \) axis.

Demo: Sugar solution

A 4’ glass cylinder filled with a supersaturated sugar solution. Polarized light (adjustable with polarizer) enter the cylinder. Because of the lack of microscopic mirror symmetry, the plane of polarization of the light is rotated by different amount, depends on the wavelength. \( \Rightarrow \) Rotational dispersion & look like a barber pole!
But... How do we produce EM waves?!!

\[ \Rightarrow \text{Radiation: from a point source} \]

In vacuum, EM wave neither lose or gain energy.

\[ \text{Poynting vector: } \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \quad \text{"rate of energy transfer/area"} \]

\[ \langle S \cdot A \rangle_1 = \langle S \cdot A \rangle_2 = \text{power} \]

\[ \langle S \rangle \propto \frac{1}{A} \propto \frac{1}{r^2} \]

\[ \Rightarrow \langle \vec{E} \rangle, \langle \vec{B} \rangle \propto \frac{1}{r} \]

Question: how do I produce radiation?

(i) Stationary charge:

\[ \begin{align*}
\vec{E} & = \frac{q}{4\pi\varepsilon_0 r^2} \hat{r} \propto \frac{1}{r^2} \\
\vec{B} & = 0 \\
\vec{S} & = 0
\end{align*} \]

Does not radiate

(ii) Charge at constant speed \( \vec{u} \):

\[ \beta = \frac{\vec{u}}{c} \]

\[ \vec{E} = \frac{q}{4\pi\varepsilon_0 r^2} \frac{1-\beta^2}{(1-\beta^2 \sin^2 \theta)^{3/2}} \hat{r} \]

\[ \vec{B} = \frac{\vec{u} \times \vec{E}}{c^2} \propto \frac{1}{r^2} \]
\[ |E| \propto \frac{1}{r^2}, \quad |B| \propto \frac{1}{r^3} \]

\[ \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \mathbf{S} \propto \frac{1}{r^4} \quad \Rightarrow \text{does not radiate} \]

(Or we can use simple argue: go to the rest frame of the charge)

Therefore: need to accelerate the charge to produce radiation.

Proof can be found in P355-360 Georgi

Or, the following geometrical argument:

\[ \text{Goal: To create a "kink" in the} \]
\[ \text{electric field line:} \]

\[ \text{Accelerated Charge?} \]

Consider a charge, accelerated between Time = 0 to \( \Delta t \)

\( \Delta t \) is small, \( \Delta t \) is small
It takes time for information to propagate (at the speed of light)

1. Surface: information that the charge accelerated has only just reached this sphere.

2. Surface: information that the charge moving with constant velocity has reached this sphere.

Q: What will ☀ see at time = t+Δt? A: A stationary charge

Therefore: outside ☀: electric field is like the charge has never moved (where ☀ lives)

Inside ☀: electric field is in the \( \hat{r} \) direction.

Between ☀ and ☀, the field must be continuous because there is no source between them.
Since \( U = a \cdot \Delta t \) is \( \ll C \)

\[ \Rightarrow \text{the velocity of the charge after acceleration} \]

\[ \Rightarrow \text{field line from A and B are} \]

\[ \Rightarrow \text{parallel} \]

\[ \Rightarrow \text{We manage to create a "kink" !!!!} \]

\[ U = a \cdot \Delta t \]

\[ E_\perp = \frac{-U_\perp}{\cot} = \frac{-a \cdot \Delta t \cdot t}{\cot} \]

\[ = \frac{-a \cdot t}{c} = \frac{-a \cdot r}{c^2} \]

\[ \Rightarrow E_\perp = \frac{-a \cdot r}{c^2} E_{\parallel} \quad (A) \]

What is \( E_{\parallel} \)?

Gauss' Law

\[ E_{\parallel} = E_{\text{ext}} = \frac{q}{4\pi\varepsilon_0 r^2} = \text{Electric field outside} \quad (B) \]

\[ \Rightarrow E_{\perp} = \frac{-9a}{4\pi\varepsilon_0 c^2 r} \quad !!! \]
Very important!!

\( E_\perp \) at position \( \vec{r} \) is due to acceleration which occurred at a retarded time:

\[ t' = t - \frac{\vec{r}}{c} \]

\[ \Rightarrow \quad E_{\text{rad}} (\vec{r}, t) = \frac{-q \vec{A}_\perp (t - \frac{\vec{r}}{c})}{4\pi \varepsilon_0 c^2 r} \]

\[ \Rightarrow \quad B_{\text{rad}} \propto \frac{1}{r} \]

\[ \Rightarrow \quad S_{\text{rad}} \propto E_{\text{rad}} \times B_{\text{rad}} \]

\[ \propto \frac{1}{r^2} \]

[Sending energy to the edge of the universe !!!]
\[ \vec{r} \] \rightarrow \text{scale of } \vec{a}(t) \text{ such that static contribution die out}

\[ \vec{E}_{\text{rad}}(\vec{r}, t) = \frac{-q \, \vec{a}(t - \frac{r}{c})}{4\pi \varepsilon_0 c^2 r} \]

\[ \vec{B}_{\text{rad}}(\vec{r}, t) = \frac{1}{c} \left( \vec{r} \times \vec{E}_{\text{rad}}(t) \right) \]

\[ \vec{S}_{\text{rad}}(\vec{r}, t) = \frac{1}{\mu_0} \vec{E}_{\text{rad}} \times \vec{B}_{\text{rad}} \]

\[ \vec{a}_\perp = \vec{a} - \frac{\vec{a} \cdot \vec{r}}{r^2} \vec{r} \quad \hat{r} = \frac{\vec{r}}{r^2} \]

1. Get \( \vec{a} \)

2. Define \( \vec{r} \), get \( \vec{a}_\perp \) \( \vec{a}_\perp = \vec{a} - \vec{a} \cdot \hat{r} \hat{r} \)

3. \( \vec{E}_{\text{rad}} \)

4. \( \vec{B}_{\text{rad}} = \frac{1}{c} \vec{r} \times \vec{E}_{\text{rad}} \)

5. \( \vec{S}_{\text{rad}} = \frac{1}{\mu_0} \vec{E}_{\text{rad}} \times \vec{B}_{\text{rad}} \)

6. Total power \( \dot{p}(t) = \iiint_{\Sigma(t)} \vec{S}_{\text{rad}} \cdot d\vec{A} \hat{n} = \frac{q^2 \, (t - \frac{r}{c})^2}{8\pi \varepsilon_0 c^3} \)
Example: harmonically oscillating charge

\[ \chi = \ddot{z} \hat{z} d \cos \omega t \]

\[ R \gg d \]

(1) At a distance \( R \) away from the charge in the \( \hat{z} \) direction:

\[ \mathbf{a}(t) = \dot{x}(t) = -\hat{z} d \omega^2 \cos \omega t \]

\[ E_{\text{rad}}(R,t) = \frac{-a \mathbf{a}_\perp(t - \frac{R}{c})}{4 \pi \varepsilon_0 c^2 R} \]

\[ \mathbf{a}_\perp = \mathbf{a} - \mathbf{a} \cdot \hat{\mathbf{R}} \hat{\mathbf{R}} \]

in this case \( \mathbf{a} \parallel \hat{z} \)

\[ \Rightarrow \mathbf{a}_\perp = 0 \]

\[ \Rightarrow \text{No radiation!} \]

(2) How about \( R \hat{\mathbf{y}} \) ?

\[ \mathbf{a}_\perp = \mathbf{a} - \mathbf{a} \cdot \hat{\mathbf{y}} \hat{\mathbf{y}} = -\hat{z} d \omega^2 \cos \omega t \]

\[ \Rightarrow \mathbf{E}(t) = \frac{+gd \omega^2 \cos \omega t - \frac{R}{c}}{4 \pi \varepsilon_0 c^2 R} \hat{z} \]

\[ \mathbf{B}_{\text{rad}}(t) = \frac{1}{c} \hat{\mathbf{R}} \times \mathbf{E}_{\text{rad}}(t) = \frac{gd \omega^2 \cos (\omega t - \frac{R}{c})}{4 \pi \varepsilon_0 c^3 R} \hat{\mathbf{x}} \]
We get harmonic waves with amplitude decreasing v.s. $R$.

(3) How about at $R\left(\frac{1}{2} \hat{Y} + \frac{\sqrt{3}}{2} \hat{Z}\right)$

(30° angle with respect to the z-axis in the y-z plane)

$$\vec{a}_\perp(t) = \vec{a} - (\vec{a} \cdot \hat{r}) \hat{r}$$

$$= -w^2 d \cos \omega t \left(\hat{z} - \frac{\sqrt{3}}{2} \left(\frac{1}{2} \hat{Y} + \frac{\sqrt{3}}{2} \hat{Z}\right)\right)$$

$$= -w^2 d \cos \omega t \left(\frac{1}{2} \hat{z} - \frac{\sqrt{3}}{2} \hat{Y}\right)$$

$$\vec{E}_{rad}(t) = \frac{q w^2 d}{8 \pi \varepsilon_0 c^2 R} \cos (\omega (t - \frac{R}{c})) \left(\frac{1}{2} \hat{z} - \frac{\sqrt{3}}{2} \hat{Y}\right)$$
Example 2: A particle with charge \( q \) is moving on an elliptical orbit

\[
\begin{align*}
\chi(t) &= \sqrt{2} A \cos(wt) \\
y(t) &= A \sin(wt)
\end{align*}
\]

What are the polarizations of the electric field seen by distant observers on the positive \( x, y, z \) axes?

\[ \Rightarrow \text{First calculate } \hat{a}(t) \]

\[ \hat{a}(t) = -\sqrt{2} A w^2 \cos(wt) \hat{x} - A w^2 \sin(wt) \hat{y} \]

(1) Observer \( R \hat{x} \)

\[ \hat{a}_x = -A w^2 \sin(wt) \hat{y} \]

\[ \hat{E}_{rad} = \frac{q w^2 A}{4\pi \epsilon_0 c^2 R} \sin\left( w(t - \frac{R}{c}) \right) \quad \text{Linearly polarized} \]

(2) \( \hat{y} \): Similarly, also linearly polarized

(3) \( \hat{z} \): Elliptically polarized