Problem Set 5

1. **Gaussians and uncertainty product saturation** [5 points]

Consider the gaussian wavefunction

\[ \psi(x) = N \exp\left(-\frac{1}{2} \frac{x^2}{a^2}\right), \]  (1)

where \( N \in \mathbb{R} \) and \( a \) is a real positive constant with units of length. The integrals

\[ \int_{-\infty}^{\infty} dx e^{-\alpha x^2 + \beta x} = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right), \quad \text{Re}(\alpha) > 0, \]

\[ \int_{-\infty}^{\infty} dx x^2 e^{-\alpha x^2} = \frac{1}{2\alpha} \int_{-\infty}^{\infty} dx e^{-\alpha x^2} \]

(a) Use the position space wavefunction (1) to calculate the uncertainties \( \Delta x \) and \( \Delta p \). Confirm that your answer saturates the Heisenberg uncertainty product

\[ \Delta x \Delta p \geq \frac{\hbar}{2}. \]

(Hints: These calculations are actually quite brief if done the right way! Using the second of the above integrals you don’t even have to determine \( N \). For the evaluation of \( \langle \hat{p}^2 \rangle \) in position space fold one factor of \( \hat{p} \) into \( \psi^* \).)

(b) Calculate the Fourier transform \( \phi(p) \) of \( \psi(x) \). Use Parseval to confirm your answer and then recalculate \( \Delta p \) using momentum space.

2. **Complex Gaussians and the uncertainty product** [10 points]

Consider the gaussian wavefunction

\[ \psi(x) = N \exp\left(-\frac{1}{2} \frac{x^2}{\Delta^2}\right), \quad \Delta \in \mathbb{C}, \quad \text{Re}(\Delta^2) > 0, \]  (1)

where \( N \) is a real normalization constant and \( \Delta \) is now a complex number: \( \Delta^* \neq \Delta \). The integrals in Problem 1 are also useful here and so is the following relation, valid for any nonzero complex number \( z \),

\[ \text{Re}\left(\frac{1}{z}\right) = \frac{\text{Re}(z)}{|z|^2} \]  (prove it!)
(a) Use the position space representation (1) of the wavefunction to calculate the uncertainties $\Delta x$ and $\Delta p$. Leave your answer in terms of $|\Delta|$ and $\text{Re}(\Delta^2)$. ($\Delta x$ will depend on both $|\Delta|$, while $\Delta p$ will depend only on $\text{Re}(\Delta^2)$).

(b) Calculate the Fourier transform $\phi(p)$ of $\psi(x)$. Use Parseval to confirm your answer and then recalculate $\Delta p$ using momentum space.

(c) We parameterize $\Delta$ using a phase $\phi_\Delta \in \mathbb{R}$ as follows

$$\Delta = |\Delta| e^{i\phi_\Delta}.$$

Calculate the product $\Delta x \Delta p$ and confirm that the answer can be put in terms of a trigonometric function of $\phi_\Delta$ and that $|\Delta|$ drops out. Is your answer reasonable for $\phi_\Delta = 0$ and for $\phi_\Delta = \frac{\pi}{2}$?

(d) Consider the free evolution of a gaussian wave packet in Problem 3 of Homework 4. What is $\Delta p$ at time equal zero? Examine the time evolution of the gaussian (from the solution!) and read the value of the time-dependent (complex) constant $\Delta^2$. Confirm that $\Delta p$, found in (a), gives a time-independent result.

3. **Exercises with a particle in a box**  [15 points]

Consider a 1D problem for a particle of mass $m$ that is free to move in the interval $x \in [0, a]$. The potential $V(x)$ is zero in this interval and infinite elsewhere. For that system consider a solution of the Schrödinger equation of the form

$$\Psi_n(x, t) = N \sin \left( \frac{n \pi}{a} x \right) e^{-i\phi_n(t)}, \quad x \in [0, a],$$

and $\Psi_n(x, t) = 0$ for $x < 0$ and $x > a$. Here $n \geq 1$ is an integer.

(a) Find the expression for the (real) phase $\phi_n(t)$ so that the above wavefunction solves the Schrödinger equation. Find the normalization constant $N$.

(b) Use $\Psi_n(x, 0)$ to calculate $\langle x \rangle$, $\langle x^2 \rangle$, and $\Delta x$.

(c) Use $\Psi_n(x, 0)$ to calculate $\langle p \rangle$, $\langle p^2 \rangle$, and $\Delta p$.

(d) Is the uncertainty inequality satisfied? Is it saturated?

(e) What answers in (b) and (c) change for $\Psi_n(x, t)$? Explain.

4. **A Hard Wall**  [5 points]

A particle of mass $m$ is moving in one dimension, subject to the potential $V(x)$:

$$V(x) = \begin{cases} 
0, & \text{for } x > 0, \\
\infty, & \text{for } x \leq 0.
\end{cases}$$

Find the stationary states and their energies. These states cannot be normalized.

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1Actually $\Delta x$ can be written in terms of $\text{Re}(1/\Delta^2)$ alone.
5. **A Step Up on the Infinite Line** [10 points]

A particle of mass $m$ is moving in one dimension, subject to the potential $V(x)$:

$$V(x) = \begin{cases} 
V_0, & \text{for } x > 0, \\
0, & \text{for } x \leq 0.
\end{cases}$$

Find the stationary states that exist for energies $0 < E < V_0$.

6. **A Wall and Half of a Finite Well** [10 points]

A particle of mass $m$ is moving in one dimension, subject to the potential $V(x)$:

$$V(x) = \begin{cases} 
\infty, & \text{for } x < 0, \\
-V_0, & \text{for } 0 < x < a, \quad (V_0 > 0) \\
0, & \text{for } x > a.
\end{cases}$$

Find the stationary states that correspond to bound states ($E < 0$, in this case). Is there always a bound state? Find the minimum value of $z_0$

$$z_0^2 = \frac{2ma^2V_0}{\hbar^2},$$

for which there are three bound states. Explain the precise relation of this problem to the problem of the finite square well of width $2a$.

7. **Mimicking hydrogen with a one-dimensional square well.** [5 points]

The hydrogen atom, the Bohr radius $a_0$ and ground state energy $E_0$ are given by

$$a_0 = \frac{\hbar^2}{me^2} \approx 0.529 \times 10^{-10} \text{ m}, \quad E_0 = -\frac{e^2}{2a_0} = -13.6 \text{ eV}.$$ 

The ground state is a bound state and the potential goes to zero at infinity. We want to design a one-dimensional finite square well

$$V(x) = \begin{cases} 
-V_0, & \text{for } |x| < a_0, \quad V_0 > 0, \\
0, & \text{for } |x| > a_0,
\end{cases}$$

that simulates the hydrogen atom. Calculate the value of $V_0$ in eV so that the ground state of the box is at the correct depth.

8. **No states with $E < V(x)$** [5 points]

Consider a real stationary state $\psi(x)$ with energy $E$:

$$-\frac{\hbar^2}{2m} \psi''(x) + [V(x) - E] \psi(x) = 0.$$ 

(a) Prove that $E$ must exceed the minimum value of $V(x)$ by noting that $E = \langle H \rangle$.

(b) Explain the claim by trying (and failing) to sketch a wavefunction consistent with being on the classically inaccessible region for all values of $x$. 