This definition in which the uncertainty of the permission operator \( Q \) in the state \( \psi \). It's always important to have a state associated with measuring the uncertainty. Because the uncertainty will be different in different states. So the state should always be there. Sometimes we write it, sometimes we get a little tired of writing it and we don't write it. But it's always implicit.

So here it is. From the analogous discussion of random variables, we were led to this definition, in which we would have the expectation value of the square of the operator minus the square of the expectation value. This was always-- well, this is always a positive quantity. Because, as claim 1 goes, it can be rewritten as the expectation value of the square of the difference between the operator and its expectation value.

This may seem a little strange. You're subtracting from an operator a number, but we know that numbers can be thought as operators as well. Operator of minus a number acting on a state is well defined. The operator acts on the state, the number multiplies a state. So this is well defined. And claim 1 is proven by direct computation.

You certainly indeed prove. You can expand what is inside the expectation value, so it's \( Q \hat{\ }^2 \) squared. And then the double product of this \( Q \hat{\ } \) and this number. Now, the number and \( Q \hat{\ } \) commute, so it is really the double product. If you have \( A + B \times A + B \), you have \( AB + BA \), but if they commute it's \( 2AB \), so this is minus 2 \( Q \hat{\ } \) \( Q \). Like that.

And then, the last term is the number squared, so it's plus \( Q \) squared. And sometimes I don't put the hats as well. And all this is the expectation value of the sum of all these things. The expectation value of a sum of things is the expectation value of the first plus the expectation value of the second, plus the expectation value of the next. So we can go ahead and do this, and this is therefore expectation value of \( Q \) squared minus the expectation value of this whole thing.

But now the expectation value of a number times an operator, the number can go out. And this is a number, and this is a number. So it's minus 2 expectation value of \( Q \), number went out. And then you're left with expectation value of another \( Q \). And the expectation value of a number is just the number, because then you're left within the world of \( \psi \star \psi \), which is equal to 1. So here is plus \( Q \hat{\ }^2 \).

And these two terms, the second and the third, are the same really. They are both equal to
expectation value of Q squared. They cancel a little bit, and they give you this. So indeed, this
is equal to expectation value of Q squared minus expectation value of Q squared.

So claim 1 is true. And claim 1 shows in particular that this number, delta Q squared, in the
expectation value of a square of something, is positive. We'll see more clearly in a second
when we have claim number 2. And claim number 2 is easily proven. That's another
expression for uncertainty.

For claim number 2, we will start with the expectation value of Q minus Q squared, like this,
which is the integral dx psi star of x and t, Q minus expectation value of Q, Q minus
expectation value of Q, on psi.

The expectation value of this thing squared is psi star, the operator, and this. And now, think of
this as an operator acting on all of that. This is a Hermitian operator. Because Q hat is
Hermitian, and expectation value of Q is real. So actually this real number multiplying
something can be moved from the wave function to the starred wave function without any cost.

So even though you might not think of a real number as a Hermitian operator, it is. And
therefore this whole thing is Hermitian. So it can be written as dx. And now you have this whole
operator, Q minus Q hat, acting on psi of x and t. And conjugate. Remember, the operator, the
Hermitian operator, moves to act on psi, and the whole thing [INAUDIBLE]. And then we have
here the other term left over.

But now, you see that you have whatever that state is and the state complex conjugated. So
that is equal to this integral. This is the integral dx of the norm squared of Q hat minus Q hat
psi of x and t squared, which means that thing, that's its complex conjugate.

So this completes our verification that these claims are true, and allow us to do the last step on
this analysis, which is to show that if you have an eigenstate of Q, if a state psi is an eigenstate
of Q, there is no uncertainty. This goes along with our measurement postulate that says an
eigenstate of Q, you measure Q and you get the eigenvalue of Q and there's no uncertainty.
In particular, we'll do it here I think.

If psi is an eigenstate of Q, so you'll have Q psi equal lambda psi, where lambda is the
eigenvalue.

Now, this is a nice thing. It's stating that the state psi is an eigenstate of Q and this is the
eigenvalue, but there is a little bit more than can be said. And it is. It should not surprise you
that the eigenvalue happens to be the expectation value of Q on the state \( \psi \). Why? Because
you can take this equation and integrate \( dx \times \psi^* \). If you bring that in into both sides of
the equation then you have \( Q \psi = \int dx \psi^* \psi \), and the lambda goes up.

Since my assumption whenever you do expectation values, your states are normalized, this is
just lambda. And by definition, this is the expectation value of Q. So lambda happens to be
equal to the expectation value of Q, so sometimes we can say that this equation really implies
that \( Q \hat{\psi} \) is equal to expectation value of \( Q \psi \) times \( \psi \).

It looks a little strange in this form. Very few people write it in this form, but it's important to
recognize that the eigenvalue is nothing else but the expectation value of the operator of that
state. But if you recognize that, you realize that the state satisfies precisely \( Q \hat{\psi} - Q \psi = 0 \). Therefore, if \( Q \hat{\psi} - Q \psi \) is equal to 0, \( \delta Q \) is equal to 0. By
claim 2, \( Q \hat{\psi} - Q \psi \) expectation value kills the state, and therefore this is 0. OK then.

The other way is also true. If \( \delta Q \) is equal to 0, by claim 2, this integral is 0. And since it's
the sum of squares that are always positive, this state must be 0 by claim 2. And you get that
\( Q \psi - Q \hat{\psi} \) equals 0. And this means that \( \psi \) is an eigenstate of \( Q \).

So the other way around it also works. So the final conclusion is \( \delta Q \) is equal to 0 is
completely equivalent of-- I'll put in the \( \psi \). \( \psi \) is an eigenstate of \( Q \).

So this is the main conclusion. Also, we learned some computational tricks. Remember you
have to compute an expectation value of a number, uncertainty, you have these various
formulas you can use. You could use the first definition. Sometimes it may be the simplest. In
particular, if the expectation value of \( Q \) is simple, it's the easiest way.

So for example, you can have a Gaussian wave function, and people ask you, what is \( \delta x \) of
the Gaussian wave function? Well, on this Gaussian wave function, you could say that
\( \delta x \) squared is the expectation value of \( x \) squared minus the expectation value of \( x \)
squared.

What is the expectation value of \( x \)? Well, it would seem reasonable that the expectation value
of \( x \) is 0. It's a Gaussian centered at the origin. And it's true. For a Gaussian it would be 0, the
expectation value of \( x \). So this term is 0. You can also see 0 because of the integral. You're
integrating \( x \) against \( \psi \) squared. \( \psi \) squared is even, \( x \) is odd with respect to \( x \) going to
minus x. So that integral is going to be 0. So in this case, the uncertainty is just the calculation
of the expectation value of $x$ squared, and that's easily done. It's a Gaussian integral.

The other good thing about this is that even though we have not proven the uncertainty
principle in all generality. We've only [? multivated ?] it. It's precise with this definition. So when
you have the delta $x$, delta $p$ is greater than or equal to \( \frac{\hbar}{2} \), these things are
computed with those definitions. And then it's precise. It's a mathematically rigorous result. It's
not just hand waving. The hand waving is good. But the precise result is more powerful.