PROFESSOR: If you tell someone solve this equation for \( l \) equals zero-- so you have this effective potential for the real equation-- solve it for \( l \) equals zero and you find some energies. You solve it for \( l \) equals 1, and you've found some energies. You solve it now for \( l \) equals two and you found some energies.

Well, we found them all together, but something extraordinary happened there was no a priori reason the system should have been so simple. It might have happened that these states would have not been aligned with the previous states. Nothing we've explained in this course predicts that this would have happened, this perfect alignment with extreme amount of degeneracy.

Because you have an \( l \) equals 2 multiplet here. That means five states. So there is degeneracy, but that's implicit in angular momentum. It has an explanation. But why would there be a degeneracy between \( l \) equals 1 solutions and \( l \) equals 2 solutions? Total mystery, actually.

And this led to all kinds of interesting discoveries that have to do with the Runge-Lenz vector, which is some conserved vector in planetary orbits. In planetary orbits, in Newton's theory, an elliptical orbit is the general solution. Nevertheless, elliptical orbits do not precess.

So you have an ellipse, it goes like that. It's not going around and rotating the ellipse at the same time. The precession of an ellipse is not allowed by Newton's theory. It's allowed by Einstein's gravity theory, and in fact, the precession of mercury was measured.

But there is no precession in Newton's theory, and there's no precession in a hydrogen atom, in a sense, as you will see. And that explains, actually in a rather interesting way-- but I'm not saying how yet-- why there is this extra degeneracy. In fact, if you had solved the problem of an infinite spherical, well, you will solve that in 805.

Infinite spherical well. Suppose there's infinite square well-- infinite spherical well. Inside a sphere of radius \( A \), the potential is 0. Outside the sphere of radius \( A \), the potential is infinite. That potential that looks so symmetric-- the \( l \) equals 0 states are like this.

The \( l \) equal 1 states are like that. The \( l \) equal 2 states are like that, and there's never any coincidence. So this coincidence between the \( l \) equals 0, \( l \) equals 2, \( l \) equals 2 is very special.
It just doesn't happen often. It's a sign of an extra symmetry.

This could only be explained because the hydrogen atom has an extra symmetry you're not aware of. So that's why this Runge-Lenz vector has to do with that extra symmetry and explains this effect. And we'll get some intuition about it today. A few more remarks to get your intuition working on the hydrogen atoms.

Z equals 1, we write the wave function. This is the most famous wave function. Pi a0 cubed e to the minus r over a0. And this is for z equals 1. n equals 1, l equals 0, m equals zero-- the complete ground state.

It's interesting to note and try to think, OK, suppose they gave you the z equals 1 answer. How do you get z different from 1? Can I just do something with this solution? Well, somehow it's written here, but I don't give here the normalization because it's impossibly complicated to write the general form for the normalization.

So how should I think of changing if somebody would have told me this is the answer for Z equals 1? How do I get to Z different from 1? And then I think of the potential, and the potential was e squared over r.

That was the potential before, V of r. And it will pass to a V of r that has minus Ze squared over r. Because now you have a nucleus with Z protons interacting with one electron, so that's how it changed. So naturally, what seems to be the change here, and you could imagine just solving it without the Z and then adding the Z, is that everywhere that you have e squared you should put Z times e squared.

And then you think of a0. a0 was h squared over me squared. We calculated that some time ago. And then if e squared goes to Ze squared, this will go to 1 over Z squared over me squared. So it will go to a0 over Z.

So you change a0 to a0 over Z. And these are the two changes. One is implicit on the other, but many times you write the formula in a mixed way. Look at that energy.

If you would have looked at this without the Z and you would have said, oh, e squared is replaced by Ze squared, you would have put a single Z. But there is a Z squared here, and it comes because 1z is here and the other z is in the a0 because a0 also has the e squared. So you have to be aware that we write these things. And this is intuitively a very nice way to write the energy, because it has the right units-- electric charge squared divided by distance. But
you could have written everything with h bar, something like that, in which case the z squared might have been less surprising.

We see here, however, the z is appearing in the right place because of the a0. So here, I was right. e to the minus zr over a0. And can I get the normalization even right? At this moment, yes.

Let's do the same change here. Pi a0 cubed, z cubed. And this must be right, because, in fact, if this wave function was normalizable-- not normalizable; was normalized-- when you do the integral, somehow the a0 did not matter, must not matter.

You checked psi squared integrated over volume is equal to 1. The a0 must be canceling here. And therefore, if it works for a0, it must work for a0 over z, and that must be a wave function. So that's fine. That's one thing you could ask.

Another thing that you could ask is, at least intuitively, why did we get this factor here? Why did we get this exponential? And that's also not mysterious at all. This comes from the differential equation. It comes up rather immediately from the differential equation.

You have minus h squared over 2m d second u dr squared, plus some sort of number-- you don't care how much-- u over r squared in the effective potential, minus some number over r times u is equal to Eu. This was your radial differential equation. And this r goes to infinity. As r goes to infinity, you get minus h squared over 2m d second u dr squared is equal to Eu, roughly. That's the key terms.

And from here, d second u dr squared is equal to minus 2 mE over h squared u. And that gives you an exponential, and the exponential must be of the right value, which we can calculate easily from the expression for the energy. So the expression for the energy gives you em equals minus z squared over 2a0 e squared, times 1 over n squared. And I can change this thing into minus z squared.

Recall what's the value of the fine structure constant. I can replace e squared from a0 to get the following thing over 2a0 times h squared over ma0. That is 1 over m squared. A little bit of manipulation. So at the end, minus 2mEn over h squared, which is what I need from the differential equation.

I must multiply by minus 2m over h squared. You see that minus the 2m over h squared, they
will disappear. So you get here \( z^2 \) over \( n^2 \) \( a_0^2 \). A little bit of manipulation.

So what are we trying to get? We want to understand immediately where this came from. And we see it. It comes from the asymptotic form of the differential equation for a solution. So I calculate the value of the right-hand side is this, and therefore this differential equation now looks like the \( du/dr \) squared equals \( z^2 \) over \( n^2 \) \( a_0^2 \) u.

Are indeed, from here, the solutions are exponentials of the square root of this, which is \( z \) over \( na_0r \). And that's a quicker derivation of a feature of the wave function. It's almost like you want to look at this wave function, and you want to say, I understand where everything comes from.

And I don't have to solve pages and pages of differential equations to see why I need this, why I need that much. I know I need this from \( r \) equals zero. I know this degree. I need it from the node theorem. Everything sort of has a reason for being there, and we should understand.