Problem 1: Dust Grains in Space

a) $\mathcal{H}$ is separable: the 6 variables are statistically independent.

\[
p(\theta_1, \theta_2, \theta_3, L_1, L_2, L_3) = \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2\pi I_1 kT}} \exp\left(-\frac{L_1^2}{2I_1 kT}\right) \frac{1}{\sqrt{2\pi I_2 kT}} \exp\left(-\frac{L_2^2}{2I_2 kT}\right) \frac{1}{\sqrt{2\pi I_3 kT}} \exp\left(-\frac{L_3^2}{2I_3 kT}\right)
\]

b) \[< L_1^2 > = < L_2^2 > = I_1 kT \quad \Rightarrow \quad < L_3^2 > = I_3 kT\]

$\Rightarrow \vec{L}$ is almost $\perp$ to axis 3, the long axis.

c) \[Z_R = (Z_{1,R})^N = \left[(2\pi)^{9/2} \sqrt{I_1 I_2 I_3 (kT)^3/2}\right]^N\]

\[F_R = -NkT \ln Z_{1,R}\]

\[S_R = -\left(\frac{\partial F_R}{\partial T}\right)_N = Nk \ln Z_{1,R} + NkT \frac{1}{Z_{1,R}} \frac{3}{2} \frac{1}{T} Z_{1,R}\]

\[= Nk \ln Z_{1,R} + \frac{3}{2} Nk\]

d) 3\textsuperscript{rd} law is violated: \[\lim_{T \to 0} S_R = Nk \ln(0) = -\infty.\] At very low temperatures one must switch to a quantum treatment of the rotational motion. Such a treatment will lead to a result consistent with the 3\textsuperscript{rd} law.

e) There is no energy gap behavior because there is no gap in the classically allowed rotational energies. The quantum result, however, will show an energy gap.
Problem 2: Adsorption On a Stepped Surface

a) $Z_1 = \sum_{\text{states}} \exp(-\epsilon_{\text{state}}/kT) = 0.01M + 0.14M \exp(-\Delta/kT) + 0.85M \exp(-1.5\Delta/kT)$

b) 
\[
\frac{n_{\text{face}}}{n_{\text{corner}}} = \frac{p_{\text{face}}}{p_{\text{corner}}} = \frac{0.85M \exp(-1.5\Delta/kT)}{0.01M} = 85 \exp(-1.5\Delta/kT)
\]

c) Consider only the 2 lowest energy levels
\[
E = N <\epsilon_{\text{one}}>
\]
\[
= N \left[ \frac{0.01M}{0.01M + 0.14M \exp(-\Delta/kT)} + \frac{0.14M \exp(-\Delta/kT)}{0.01M + 0.14M \exp(-\Delta/kT)} \right] 
\approx 14N\Delta \exp(-\Delta/kT)
\]

\[
C_A = \left( \frac{\partial E}{\partial T} \right)_A = 14N\Delta \left( \frac{\Delta}{kT^2} \right) \exp(-\Delta/kT) = 14Nk \left( \frac{\Delta}{kT} \right)^2 \exp(-\Delta/kT)
\]

d) All states are equally likely $\Rightarrow p_{\text{face}} = 0.85$.

e) $M$ possible states for each atom $\Rightarrow \lim_{T \to \infty} S = Nk \ln M$.

f) One expects energy gap behavior because there is an energy gap for the excitation of a single atom.
**Problem 3: Neutral Atom Trap**

a) First write down the Hamiltonian for one atom.

\[ H_1 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + ar \]

Then compute the partition function

\[
Z_1 = \frac{1}{\hbar^3} \int_{-\infty}^{\infty} e^{-\frac{p_x^2}{2mkT}} dp_x \int_{-\infty}^{\infty} e^{-\frac{p_y^2}{2mkT}} dp_y \int_{-\infty}^{\infty} e^{-\frac{p_z^2}{2mkT}} dp_z \int_0^{2\pi} r^2 \sin \theta dr d\theta d\phi \\
= (\frac{2\pi mkT}{\hbar^2})^{3/2} 4\pi \left( \frac{kT}{a} \right)^3 \int_0^\infty y^2 e^{-y} dy \\
= 8\pi k^3 (\frac{2\pi mk}{\hbar^2})^{3/2} T^{9/2} a^{-3}
\]

In order to emphasize the dependence on the important variables, this can be written in the form

\[ Z_1 = A T^\alpha a^{-\eta} \]

where

\[ A = 8\pi k^3 (\frac{2\pi mk}{\hbar^2})^{3/2} \]

\[ \alpha = 9/2 \quad \text{and} \quad \eta = 3. \]

b) Remember to include correct Boltzmann counting.

\[
Z = \frac{1}{N!} Z_1^N \\
F = -kT \ln Z = -kT(N \ln Z_1 - N \ln N + N) \\
= -NkT \ln(Z_1/N) - NkT \\
S = -\left( \frac{\partial F}{\partial T} \right)_N \\
= Nk \ln(Z_1/N) + Nk + NkT \frac{1}{Z_1/N} (9/2) \frac{Z_1/N}{T} \\
= Nk \ln(Z_1/N) + (11/2)Nk
\]
c) \( dQ = 0 \) no heat is exchanged with surroundings

\[ dQ = \frac{dS}{T} \quad \text{process is said to be reversible} \]

\[ \Rightarrow dS = 0, \quad S \text{ is constant} \]

\[ \Rightarrow Z_1 \text{ is constant, using the result from b)} \]

\[ \Rightarrow T_0^{9/2}/a^3 \text{ is constant and } = T_0^{9/2}/a_0^3 \]

\[ \left( \frac{T}{T_0} \right)^{9/2} = \left( \frac{a}{a_0} \right)^3 \]

\[ T = T_0 \left( \frac{a}{a_0} \right)^{2/3} \]

**Problem 4** Two-Dimensional H\(_2\) Gas

a)

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \epsilon )</th>
<th>DEGENERACY</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pm 3 )</td>
<td>( 9h^2/2I )</td>
<td>2</td>
</tr>
<tr>
<td>( \pm 2 )</td>
<td>( 4h^2/2I )</td>
<td>2</td>
</tr>
<tr>
<td>( \pm 1 )</td>
<td>( h^2/2I )</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ \epsilon_m = (h^2/2I) m^2 \]

\[ Z_{\text{ROT,1}} = \sum_{\text{states}} \exp[-\epsilon(\text{state})/kT] = \sum_{m=-\infty}^{\infty} \exp[-(h^2/2IkT) m^2] \]

\[ = 1 + 2 \sum_{j=1}^{\infty} \exp[-(h^2/2IkT) j^2] \]
b) \[ p(m = 3) \over p(m = 2) = \frac{Z^{-1}\exp[-9\hbar^2/2IkT]}{Z^{-1}\exp[-4\hbar^2/2IkT]} = \exp[-(5/2)\hbar^2/IkT] \]

c) \[ p(m = 3 \mid \epsilon = 9\hbar^2/2I) = \frac{Z^{-1}\exp[-9\hbar^2/2IkT]}{2(Z^{-1}\exp[-9\hbar^2/2IkT])} = 1/2 \]
\[ p(m = 1 \mid \epsilon \leq \hbar^2/2I) = \frac{Z^{-1}\exp[-\hbar^2/2IkT]}{Z^{-1}+2(Z^{-1}\exp[-\hbar^2/2IkT])} = \frac{1}{2 + \exp[\hbar^2/2IkT]} \]

d) \[ Z_{\text{ROT},1} = \sum_{m=-\infty}^{\infty} \exp[-(\hbar^2/2IkT)m^2] \rightarrow \int_{-\infty}^{\infty} \exp[-(\hbar^2/2IkT)m^2] \, dm \]
\[ = \int_{-\infty}^{\infty} \exp[-\frac{m^2}{2(IkT/\hbar^2)}] \, dm = \sqrt{\frac{2\pi IkT}{\hbar^2}} \sim \beta^{-1/2} \quad \text{Gaussian normalization} \]
\[ E_{\text{ROT}} = N <\epsilon> = N \left( -\frac{1}{Z_1} \frac{\partial Z_1}{\partial \beta} \right) \]
\[ = N \left( -\frac{1}{Z_1} \right) \left( -\frac{Z_1}{2\beta} \right) = (1/2)N \frac{1}{\beta} = (1/2)NkT \]
Problem 5: Why Stars Shine

a) The electrostatic potential outside the charged sphere depends only on $r$, the magnitude of the distance from the center of the sphere.

$$\phi(r) = \frac{|e|}{r} \quad r \geq R$$

The potential energy of another proton, considered to be a point particle, in this field is

$$V(r) = q\phi(r) = \frac{e^2}{r}$$

Then the minimum energy that the second proton must have to get within a radial distance $R$ of the first is

$$E_{\text{min}} = V(R) = \frac{e^2}{R} = \frac{(4.8 \times 10^{-10})^2}{1.2 \times 10^{-13}} = 1.92 \times 10^{-6} \text{ ergs}$$

b) In problem 4 of problem set 2 we found the following expression for the kinetic energy of a particle in a three dimensional classical gas.

$$p(E) = \frac{2}{\sqrt{\pi} kT} \sqrt{\frac{E}{kT}} \exp[-E/kT]$$

Now find the probability $p_+$ that a given proton in the stellar plasma has an energy greater than $E_{\text{min}}$.

$$p_+ \equiv \text{prob}(E > E_{\text{min}}) = \int_{E_{\text{min}}}^{\infty} p(E) \, dE$$

$$= \frac{2}{\sqrt{\pi} kT} \int_{E_{\text{min}}}^{\infty} \sqrt{\frac{E}{kT}} \exp[-E/kT] \, dE = \frac{2}{\sqrt{\pi}} \int_{y_{\text{min}} = E_{\text{min}}/kT}^{\infty} \sqrt{y} \exp[-y] \, dy$$

$$\approx \frac{2}{\sqrt{\pi}} \sqrt{y_{\text{min}}} \exp[-y_{\text{min}}]$$

This is going to turn out to be a very small number, probably too small to be represented on a hand calculator. Therefore, let’s work toward getting its logarithm.

$$\log_{10}(p_+) = \log_{10} \left[ \frac{2}{\sqrt{\pi}} \sqrt{\frac{E_{\text{min}}}{kT}} \right] + \log_{10} \left[ \exp[-E_{\text{min}}/kT] \right]$$

$$\log_{10} \left[ \exp[-E_{\text{min}}/kT] \right] = \frac{E_{\text{min}}}{kT} \log_{10}(e) = -0.4343 \frac{E_{\text{min}}}{kT}$$

$$\frac{E_{\text{min}}}{kT} = \frac{1.920 \times 10^{-6}}{1.381 \times 10^{-16} \times 4 \times 10^7} = 3.476 \times 10^2$$

$$\log_{10}(p_+) = 1.323 - 1.510 \times 10^2 = -149.6$$

$$p_+ = 0.2 \times 10^{-149}$$
c)  
\[
<v> = \sqrt{\frac{8kT}{\pi m}}
\]
\[
= \left( \frac{8 \times 1.381 \times 10^{-16} \times 4 \times 10^7}{\pi \times 1.67 \times 10^{-24}} \right)^{1/2}
= 9.18 \times 10^7 \text{ cm/sec}
\]
\[
\sigma = \pi(2R)^2 = \pi(2.4 \times 10^{-13})^2 = 1.81 \times 10^{-25} \text{ cm}^2
\]
\[
n = \frac{\rho}{M_{\text{proton}}} = \frac{100}{1.67 \times 10^{-24}} = 5.99 \times 10^{25} \text{ protons/cm}^3
\]
\[
L = (n\sigma)^{-1} = 9.22 \times 10^{-2} \text{ cm}
\]
\[
\tau_{\text{collision}} = L/ <v> = 1.01 \times 10^{-9} \text{ sec}
\]

d) The fusion rate per proton is \( p_+ \) times the collision rate per proton. But in general a rate equals the reciprocal of the characteristic time between events, so
\[
\tau_{\text{fusion}} = \tau_{\text{collision}}/p_+ = \frac{1.01 \times 10^{-9}}{0.2 \times 10^{-149}} = 5 \times 10^{140} \text{ sec}
\]
The universe is about 15 billion years old, corresponding to a time
\[
T_{\text{universe}} = 15 \times 10^9 \times 365 \times 24 \times 60 \times 60 = 4.7 \times 10^{17} \text{ sec}
\]
If the mass of the sun is \( 2 \times 10^{33} \) grams then the number of protons it contains is given by
\[
N_{\text{protons}} = \frac{2 \times 10^{33}}{1.67 \times 10^{-24}} = 1.2 \times 10^{57}
\]
Then for the entire sun, the total number of fusions per second is found as follows.
\[
\text{number of fusions per second} = N_{\text{protons}} \times \text{fusion rate per proton}
\]
\[
= N_{\text{protons}}/\tau_{\text{fusion}}
\]
\[
= 1.2 \times 10^{57} / 5 \times 10^{140} = 2 \times 10^{-84} \text{ sec}^{-1}
\]