Problem 1 (30 points) Doping a Semiconductor

When diffusing impurities into a particular semiconductor the probability density \( p(x) \) for finding the impurity a distance \( x \) below the surface is given by

\[
p(x) = \begin{cases} 
  (0.8/l) \exp[-x/l] + 0.2 \delta(x - d) & x \geq 0 \\
  0 & x < 0 
\end{cases}
\]

where \( l \) and \( d \) are parameters with the units of distance. The delta function arises because a fraction of the impurities become trapped on an accidental grain boundary a distance \( d \) below the surface.

a) Make a carefully labeled sketch of the cumulative function \( P(x) \) which displays all of its important features. [You do not need to give an analytic expression for \( P(x) \).]

b) Find \( <x> \).

c) Find the variance of \( x \), \( \text{Var}(x) \equiv <(x - <x>)^2> \).

The contribution to the microwave surface impedance due to an impurity decreases exponentially with its distance below the surface as \( e^{-x/s} \). The parameter \( s \), the “skin depth”, has the units of distance.

d) Find \( <e^{-x/s}> \).
Problem 2 (40 points) Collision Products

A certain collision process in high energy physics produces a number of biproducts. When the biproducts include a pair of elementary particles \( A \) and \( B \) the energies of those particles, \( E_A \) and \( E_B \), are distributed according to the joint probability density

\[
p(E_A, E_B) = \frac{4E_B(E_A - E_B)}{\Delta^4} \exp\left[-\frac{(E_A + E_B)}{\Delta}\right] \quad \text{for} \quad E_A > 0 \quad \text{and} \quad E_A > E_B > 0
\]

\[
= 0 \quad \text{elsewhere}
\]

\( \Delta \) is a parameter with the units of energy. A contour plot of \( p(E_A, E_B) \) is shown above. Note that the energy \( E_A \) is always positive and greater than the energy \( E_B \).

a) Find \( p(E_B) \). Sketch the result.

b) Find the conditional probability density \( p(E_A \mid E_B) \). Sketch the result.

c) Are \( E_A \) and \( E_B \) statistically independent? Explain your reasoning.

The collisions are statistically independent random events that occur at some uniform rate in time. The pair \( A \) and \( B \) only occurs in a fraction \( f \) of the collisions. When the pair is produced, it is detected with 100\% efficiency. When the pair is not produced, there are no competing background events.

d) If the overall collision rate is \( 10^6 \) per hour, how long must one run the experiment in order that the uncertainty in the determination of \( f \) is of the order of one part in \( 10^4 \) of the value of \( f \) measured in that run? Note: one does not need the answers to a), b), or c) to answer this question.
Problem 3 (30 points) Equipment Failure

A graduate student begins an experiment which depends on two critical pieces of apparatus: a dilution refrigerator and a sophisticated laser system. Each is prone to failure, the failures are statistically independent, and a failure of either one ends the experimental run. The probability of failure after a time $t$ for the refrigerator is given by

$$p(t_r) = \frac{1}{\alpha} \exp\left[-\frac{t_r}{\alpha}\right] \quad t_r \geq 0$$

$$= 0 \quad t_r < 0$$

and for the laser by

$$p(t_l) = \frac{1}{\beta} \exp\left[-\frac{t_l}{\beta}\right] \quad t_l \geq 0$$

$$= 0 \quad t_l < 0$$

We want to find the probability density for the duration of an experimental run $T$; that is, we want to find the probability density for $T \equiv \text{Min}(t_r, t_l)$.

a) Find an analytic expression for the cumulative function $P(T)$. No short cuts here; do the integrals. [Hint: a bit of thought beforehand can decrease the work considerably.]

b) Find the probability density $p(T)$ and sketch the result.
Integrals

$$\int e^{ax} \, dx = \frac{e^{ax}}{a}$$

$$\int x e^{ax} \, dx = \frac{e^{ax}}{a^2} (ax - 1)$$

$$\int x^2 e^{ax} \, dx = \frac{e^{ax}}{a^3} (a^2 x^2 - 2ax + 2)$$

$$\int \frac{dx}{1 + e^x} = \ln \left[ \frac{e^x}{1 + e^x} \right]$$

Definite Integrals

For integer $n$ and $m$

$$\int_0^\infty x^n e^{-x} \, dx = n!$$

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}} \, dx = \sqrt{\pi}$$

$$(2\pi \sigma^2)^{-1/2} \int_{-\infty}^\infty x^{2n} e^{-x^2/2\sigma^2} \, dx = 1 \cdot 3 \cdot 5 \cdots (2n - 1) \sigma^n$$

$$\int_0^\infty x e^{-x^2} \, dx = \frac{1}{2}$$

$$\int_0^1 x^m (1 - x)^n \, dx = \frac{n! m!}{(m + n + 1)!}$$