Problem 1 (25 points) Bose Gas

In a weakly interacting gas of Bose particles at low temperature the expansion coefficient $\alpha$ and the isothermal compressibility $K_T$ are given by

$$\alpha \equiv \frac{1}{V} \frac{\partial V}{\partial T} \bigg|_p = \frac{5}{4} a \frac{T^{3/2}}{c} V^2 + \frac{3}{2} b \frac{T^2}{c} V^2$$

$$K_T \equiv -\frac{1}{V} \frac{\partial V}{\partial P} \bigg|_T = \frac{1}{2c} V^2$$

where $a$, $b$ and $c$ are constants. It is known that the pressure goes to zero in the limit of large volume and low temperature. Find the equation of state $P(T, V)$.

Problem 2 (35 points) Hydrostatic System

The internal energy $U$ of a certain hydrostatic system is given by

$$U = AP^2V$$

where the constant $A$ has the units of (pressure)$^{-1}$.

a) Find the slope, $dP/dV$, of an adiabatic path ($dQ = 0$) in the $P$-$V$ plane in terms of $A$, $P$ and $V$.

Assume that one also knows the thermal expansion coefficient $\alpha$ and the isothermal compressibility $K_T$.

$$\alpha \equiv \frac{1}{V} \frac{\partial V}{\partial T} \bigg|_p \quad \text{and} \quad K_T \equiv -\frac{1}{V} \frac{\partial V}{\partial P} \bigg|_T$$

b) Find the slope, $dP/dV$, of an isothermal path in the $P$-$V$ plane.

c) Find the constant volume heat capacity, $C_V$, in terms of the known quantities.
Problem 3 (40 points) Molecular Solid

In a particular molecular solid the individual molecules are localized at specific lattice sites and possess no center of mass motion. However, each of the $N$ molecules is free to rotate about a fixed direction in space which we will designate as the $z$ direction. As far as the rotational motion is concerned the molecules can be considered to be non-interacting. The classical microscopic state of each molecule is specified by a rotation angle $0 \leq \theta < 2\pi$ and a canonically conjugate angular momentum $-\infty < l < \infty$ about the $z$ axis. The energy of a single molecule is independent of $\theta$ and depends quadratically on $l$. Thus the Hamiltonian for the system is given by

$$H = \sum_{i=1}^{N} \frac{l_i^2}{2I}$$

where $I$ is the moment of inertia of a molecule about the $z$ axis.

a) Represent the system by a microcanonical ensemble where the energy lies between $E$ and $E + \Delta$. Find an expression for the phase space volume $\Omega$. Use Sterling’s approximation to simplify your result. [It may be helpful to consult the attached information sheet.]

b) Based on your calculations in a) find the probability density $p(\theta)$ for the orientation angle of a single molecule and explain your method.

c) The probability density $p(l)$ for the angular momentum of a single molecule can be written in the form $p(l) = \Omega' / \Omega$ where $\Omega = \Omega(E, N)$ is the quantity you found in a). Find $\Omega'$. Do not try to simplify your answer. Do explain how to eliminate $E$ from your expression for $p(l)$.

d) Find the energy of the system as a function of temperature, $E(T, N)$. 

2
PARTIAL DERIVATIVE RELATIONSHIPS

Let \( x, y, z \) be quantities satisfying a functional relation \( f(x, y, z) = 0 \). Let \( w \) be a function of any two of \( x, y, z \). Then

\[
\left( \frac{\partial x}{\partial y} \right)_w \left( \frac{\partial y}{\partial z} \right)_w = \left( \frac{\partial x}{\partial z} \right)_w
\]

\[
\left( \frac{\partial x}{\partial y} \right)_z \left( \frac{\partial y}{\partial z} \right)_x = -1
\]

COMBINATORIAL FACTS

There are \( K! \) different orderings of \( K \) objects. The number of ways of choosing \( L \) objects from a set of \( K \) objects is

\[
\frac{K!}{(K-L)!}
\]

if the order in which they are chosen matters, and

\[
\frac{K!}{L!(K-L)!}
\]

if order does not matter.

STERLING’S APPROXIMATION

When \( K \gg 1 \)

\[
\ln K! \approx K \ln K - K \quad \text{or} \quad K! \approx (K/e)^K
\]

DERIVATIVE OF A LOG

\[
\frac{d}{dx} \ln u(x) = \frac{1}{u(x)} \frac{du(x)}{dx}
\]

VOLUME OF AN \( \alpha \) DIMENSIONAL SPHERE OF RADIUS \( R \)

\[
\frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)} R^\alpha
\]

LIMITS

\[
\lim_{{n \to \infty}} \frac{\ln n}{n} = 0
\]

\[
\lim_{{n \to \infty}} \sqrt{n} = 1
\]

\[
\lim_{{n \to \infty}} x^{1/n} = 1 \quad (x > 0)
\]

\[
\lim_{{n \to \infty}} x^n = 0 \quad (|x| < 1)
\]

\[
\lim_{{n \to \infty}} \left( 1 + \frac{x}{n} \right)^n = e^x \quad \text{(any } x \text{)}
\]

\[
\lim_{{n \to \infty}} \frac{x^n}{n!} = 0 \quad \text{(any } x \text{)}
\]

WORK IN SIMPLE SYSTEMS

<table>
<thead>
<tr>
<th>System</th>
<th>Intensive quantity</th>
<th>Extensive quantity</th>
<th>Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrostatic system</td>
<td>( P )</td>
<td>( V )</td>
<td>(-PdV)</td>
</tr>
<tr>
<td>Wire</td>
<td>( F )</td>
<td>( F )</td>
<td>( FdL )</td>
</tr>
<tr>
<td>Surface</td>
<td>( S )</td>
<td>( A )</td>
<td>( SdA )</td>
</tr>
<tr>
<td>Reversible cell</td>
<td>( E )</td>
<td>( Z )</td>
<td>( EdZ )</td>
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<tr>
<td>Dielectric material</td>
<td>( E )</td>
<td>( P )</td>
<td>( EdP )</td>
</tr>
<tr>
<td>Magnetic material</td>
<td>( H )</td>
<td>( M )</td>
<td>( HdM )</td>
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