Problem 1 (25 points) Polar Molecules

In a particular situation polar molecules (molecules possessing a permanent electric dipole moment) can be adsorbed on a surface creating a dipole layer with a total electric dipole moment $\mathcal{P}$ that remains finite even when the electric field perpendicular to the surface $\mathcal{E}$ goes to zero. Expressions for two important response functions in this system are given below.

$$\chi_T \equiv \left. \frac{\partial \mathcal{P}}{\partial \mathcal{E}} \right|_T = \left( a + \frac{b}{T} \right) N + 3cN\mathcal{E}^2$$

$$\left. \frac{\partial T}{\partial \mathcal{E}} \right|_{\mathcal{P}} = \frac{aT^2 + bT + 3cT^2\mathcal{E}^2}{b\mathcal{E} - dT^2}$$

In these expressions $a$, $b$, $c$ and $d$ are constants and $N$ is the number of molecules. One also knows that $\mathcal{P} = \mathcal{P}_0$ when $T = T_0$ and $\mathcal{E} = 0$. Find an analytic expression for the electric dipole moment $\mathcal{P}$. 
Problem 2 (40 points) Elastic Rod

The internal energy $U$ and the tension $F$ in a certain elastic rod are given by the expressions

$$U(T, L) = \frac{cT^4}{4} + \frac{a}{2}(L - L_0)^2$$

$$F(T, L) = (a + bT)(L - L_0)$$

where $a$, $b$, $c$ and $L_0$ are constants.

a) Find the work done on the rod, $\Delta W$, as its length is doubled from $L_0$ to $2L_0$ along an isotherm at temperature $T$.

b) Find the heat added to the rod, $\Delta Q$, along the same path as in a).

c) Find the differential equation $\frac{dL}{dT} = f(L, T)$ governing an adiabatic path in the $L - T$ plane. [Hint: you may want to check to see if your result is consistent with the sketch given above.]
Problem 3 (35 points) One-dimensional Ising Model

N spins are equally spaced around a circle in the $x$-$y$ plane. Each spin can point either parallel or antiparallel to the $z$ direction. There is no applied magnetic field so neither orientation is preferred. However, the spins interact with each other through a nearest neighbor interaction. If two neighboring spins point in the same direction, they contribute an amount $-J$ to the total energy; if they point in opposite directions, they contribute an amount $J$. Thus the total energy of the system depends on the number of reversals, $R$, that occur around the ring.

$$E = JR - J(N - R) = J(2R - N)$$

a) Assume that $N$ is even. What are the smallest and largest values that $R$ can have? What are the minimum and maximum values of $E$?

b) Find the total number of microscopic states of the system consistent with a given number of reversals, $\Omega(R)$. Note that this corresponds to the number of ways the $R$ reversals can be distributed among the $N$ inter-spin locations.

c) Assume that $N$ is large. Find the entropy $S$ of the spin chain as a function of $N$ and $R$.

d) Find the energy of the spin chain as a function of temperature, $E(T)$. Make a sketch of the resulting function for the case $J > 0$ and indicate the low and high temperature asymptotes. Consider only positive $T$.

e) What is the mean value of $R$ in the high temperature limit?
PARTIAL DERIVATIVE RELATIONSHIPS

Let \( x, y, z \) be quantities satisfying a functional relation \( f(x, y, z) = 0 \). Let \( w \) be a function of any two of \( x, y, z \). Then

\[
\left( \frac{\partial x}{\partial y} \right)_w \left( \frac{\partial y}{\partial z} \right)_w = \left( \frac{\partial x}{\partial z} \right)_w
\]

\[
\frac{\partial x}{\partial y} = \frac{1}{\left( \frac{\partial y}{\partial x} \right)_z}
\]

\[
\left( \frac{\partial x}{\partial y} \right)_z \left( \frac{\partial y}{\partial z} \right)_x \left( \frac{\partial z}{\partial x} \right)_y = -1
\]

COMBINATORIAL FACTS

There are \( K! \) different orderings of \( K \) objects. The number of ways of choosing \( L \) objects from a set of \( K \) objects is

\[
\frac{K!}{(K-L)!}
\]

if the order in which they are chosen matters, and

\[
\frac{K!}{L!(K-L)!}
\]

if order does not matter.

STERLING’S APPROXIMATION

\[
\ln K! \approx K \ln K - K \quad \text{when} \quad K \gg 1
\]

DERIVATIVE OF A LOG

\[
\frac{d}{dx} \ln u(x) = \frac{1}{u(x)} \frac{du(x)}{dx}
\]

LIMITS

\[
\lim_{n \to \infty} \frac{\ln n}{n} = 0
\]

\[
\lim_{n \to \infty} \sqrt[n]{n} = 1
\]

\[
\lim_{n \to \infty} x^{1/n} = 1 \quad (x > 0)
\]

\[
\lim_{n \to \infty} x^n = 0 \quad (|x| < 1)
\]

\[
\lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x \quad \text{(any } x)\]

\[
\lim_{n \to \infty} \frac{x^n}{n!} = 0 \quad \text{(any } x)\]

WORK IN SIMPLE SYSTEMS

<table>
<thead>
<tr>
<th>System</th>
<th>Intensive quantity</th>
<th>Extensive quantity</th>
<th>Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrostatic system</td>
<td>( P )</td>
<td>( V )</td>
<td>( -PdV )</td>
</tr>
<tr>
<td>Wire</td>
<td>( \mathcal{F} )</td>
<td>( L )</td>
<td>( \mathcal{F}dL )</td>
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<tr>
<td>Surface</td>
<td>( S )</td>
<td>( A )</td>
<td>( SdA )</td>
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<tr>
<td>Reversible cell</td>
<td>( E )</td>
<td>( Z )</td>
<td>( EdZ )</td>
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<tr>
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<td>( \varepsilon )</td>
<td>( \mathcal{P} )</td>
<td>( \varepsilon d\mathcal{P} )</td>
</tr>
<tr>
<td>Magnetic material</td>
<td>( H )</td>
<td>( M )</td>
<td>( HdM )</td>
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