Announcements

• Please put your name and section number at the top of your problem set, and place it in the 8.05 box labeled with your section number near 8-395 by 3pm Friday.

• Recommended Reading for the first week: Shankar, sections 5.2, 5.3, and 5.6. Griffiths, sections 2.1, 2.2, 2.5, and 2.6.

Problem Set 1

1. Properties of a wavefunction. [10 points]

A particle of mass \( m \) in a one-dimensional potential \( V(x) \) has the wave function

\[
\psi(x) = N x \exp \left( -\frac{1}{2} \alpha x^2 \right), \quad \alpha > 0.
\]

(a) Normalize \( \psi(x) \) to determine \( N \). What is \( \langle \hat{x} \rangle \)? What is \( \langle \hat{x}^2 \rangle \)?

(b) What is \( \langle \hat{p} \rangle \)? What is \( \langle \hat{p}^2 \rangle \)?

(c) Is \( \psi(x) \) a position eigenstate? Is \( \psi(x) \) a momentum eigenstate? Explain.

(d) Suppose that \( V(x) = 0 \). What is \( \langle \hat{H} \rangle \)?

(e) Suppose that nothing is known about \( V(x) \), but \( \psi(x) \) is an energy eigenstate. Find the potential \( V(x) \) and the energy eigenvalue \( E \), assuming \( V(0) = 0 \). Could \( \psi(x) \) be the ground state wavefunction for the particle?

2. Energy must exceed the minimum value of the potential. [5 points]

Consider the time-independent Schrödinger equation for a particle of energy \( E \) in a potential \( V(x) \), with \( x \in (-\infty, \infty) \):

\[
\frac{d^2 \psi}{dx^2} = \frac{2m}{\hbar^2} [V(x) - E] \psi(x).
\]

\[1\] A variation on Griffiths 2.2.
Without loss of generality one can assume that $\psi(x)$ is real. Assume the potential is bounded below,

$$V(x) \geq V_{\text{min}}, \text{ for all } x,$$

where $V_{\text{min}}$ is the minimum value of the potential.

Prove that $E > V_{\text{min}}$ for normalizable solutions to exist. To do this, assume $E \leq V_{\text{min}}$ and try using equation (1) and integration to reach a clear contradiction.

3. **Three Delta Functions** [15 points]

A particle of mass $m$ moves in one dimension, subject to a potential energy function $V(x)$ which is the sum of three evenly spaced attractive delta functions:

$$V(x) = -V_0 a \sum_{n=-1}^{1} \delta(x - na) , \text{ where } V_0 > 0, \ a > 0 \text{ are constants.}$$

\begin{center}
\begin{tikzpicture}
\draw[->] (-4,0) -- (4,0) node[below] {$x$};
\draw[->] (0,-2) -- (0,2) node[right] {$V$};
\draw (-3,0) -- (-3,-2);
\draw (0,0) -- (0,-2);
\draw (3,0) -- (3,-2);
\filldraw[black] (-3,0) circle (1pt);
\filldraw[black] (0,0) circle (1pt);
\filldraw[black] (3,0) circle (1pt);
\end{tikzpicture}
\end{center}

(a) Calculate the discontinuity in the first derivative of the wavefunction at $x = -a, 0, \text{ and } a$.

(b) Consider the possible number and locations of nodes in bound state wavefunctions for this system.

(i) How many nodes are possible in the region $x > a$?

(ii) How many nodes are possible in the region $0 < x < a$?

(iii) Can there be a node at $x = a$?

(iv) Can there be a node at $x = 0$?

(c) For arbitrarily large $V_0$, how many bound states are there? Sketch them qualitatively.

(d) Derive the equation that determines the energy for the lowest energy antisymmetric bound state. Find the minimum value of $V_0$ for the bound state to exist.
4. **Estimates on the finite square well** [10 points]

Consider the finite square well potential in section 2.6 of Griffiths:

\[ V(x) = -V_0 \quad \text{for} \quad -a \leq x \leq a, \quad \text{and} \quad V(x) = 0 \quad \text{for} \quad |x| > a. \]

(a) **Number of bound states for deep well.** Assume that the well is sufficiently deep and/or wide so that \( z_0 \), defined as

\[ z_0 \equiv \frac{a}{\hbar} \sqrt{2mV_0}, \]

is a large number. Find an estimate for the number of bound states in this well using the result that the \( k \)-th bound state has \( k - 1 \) nodes. Confirm that your result is a good approximation by comparing with Figure 2.18 in the book.

(b) **Energy of the bound state for a shallow well.** Assume now that the potential is very shallow and/or narrow so that \( z_0 \) is a very small number and as a result there is just one bound state. Use the relevant equations of the problem (see Griffiths) to estimate the energy \( E \) of this state in terms of \( V_0 \) and \( z_0 \) (i.e. find the leading term of the energy in the expansion in terms of \( z_0 \), as \( z_0 \to 0 \)).

5. **Expectation value \( \langle \hat{p} \rangle \) of the momentum.** [5 points]

(a) A particle’s coordinate space wavefunction is square-integrable and real up to an arbitrary multiplicative phase:

\[ \psi(x) = e^{i\alpha} \phi(x), \]

with \( \alpha \) real and constant and \( \phi(x) \) real. Prove that the expectation value of the momentum is zero.

(b) Consider instead the wavefunction

\[ \psi(x) = \phi_1(x) + e^{i\alpha} \phi_2(x), \]

where \( \phi_1(x) \) and \( \phi_2(x) \) are each real and square-integrable. What is \( \langle \hat{p} \rangle \)? The answer can be expressed as a function of \( \alpha \) times an integral that involves the functions \( \phi_2 \) and \( d\phi_1/dx \) (or \( \phi_1 \) and \( d\phi_2/dx \)). For what values of \( \alpha \) can we be sure that \( \langle \hat{p} \rangle \) is zero without having further information about \( \phi_1 \) and \( \phi_2 \)?

(c) Consider this time the wavefunction

\[ \psi(x) = e^{ikx} \phi(x), \]

with \( k \) real and constant and \( \phi(x) \) real. Calculate \( \langle \hat{p} \rangle \).
6. **Conserved probability current.** [10 points]

Suppose \( \Psi(x, t) \) obeys the one-dimensional Schrödinger equation,

\[
-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x, t) + V(x)\Psi(x, t) = i\hbar \frac{\partial}{\partial t} \Psi(x, t).
\]

(a) Derive the conservation law for probability,

\[
\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0,
\]

where \( \rho(x, t) \) is the probability density and \( J(x, t) \) is the probability current density

\[
\rho(x, t) = \Psi^*\Psi, \quad J(x, t) = \frac{\hbar}{m} \text{Im} \left( \Psi^* \frac{\partial \Psi}{\partial x} \right).
\]

What are the units of \( \rho \) and \( J \)?

(b) Explain why (3) is a conservation law for probability. In order to do so, define

\[
P_{ab}(t) \equiv \int_a^b dx \rho(x, t),
\]

evaluate \( \frac{dP_{ab}}{dt} \) in terms of currents, and interpret your answer. Show then that a wavefunction \( \Psi(x, t) \) that is normalized at time \( t \) remains normalized at later times.

(c) In the following we consider stationary states with spatial wavefunctions \( \psi(x) \).

Compute the probability current \( J \) for \( \psi(x) = e^{i\alpha(x)}\phi(x) \) where \( \alpha(x) \) and \( \phi(x) \) are real. Show that

\[
\frac{J(x)}{\rho(x)} = \frac{\hbar}{m} \alpha'(x).
\]

Explain why the ratio \( J/\rho \) can be viewed as the local velocity of the quantum particle described by \( \psi(x) \).

(d) Consider \( \psi(x) = Ae^{ipx/\hbar} + Be^{-ipx/\hbar} \), with \( A \) and \( B \) complex constants. Calculate \( J(x) \). Are there cross terms in \( J \) between the left and right-moving parts of \( \psi \)?

7. **Griffiths Problem 2.38, p.85** [10 points]