So one more thing let's do with this operator. So we're getting accustomed to these operators, and these permutation operators can also act on operators themselves. So that's important. So consider the action on operators.

So for example, an operator $B$ that acts belonging to the linear operator some $V$, when acting on $V$ tensor $V$, we define two operators, $B_1$ and $B_2$. And you define them in an obvious way, like $B_1$ acting on $U_i^1$ tensor $U_j^2$.

OK, $B$ as an operator knows how to act on every vector on the vector space capital $V$. So when you say 1, you're meaning that this operator acts on the first Hilbert space. So this is equal to $B$ times $U_i^1$ tensor $U_j^2$. So it just acts on the first state. How does it act? Via $B$, that is an operator, in the vector space $V$.

Similarly, if you have $B_2$ of $U_i$ tensor $U_j^2$ you have $U_i^1$ tensor $BU_j^2$ to OK. So these are operators that act either on the first state or in the second state. So the permutation operators can do things to these operators, as well.

So we can ask a question, what is $P_{21}B$-- should I start with one? Yes, one-- $P_{21}$ dagger. Remember, when you ask how an operator acts on an operator, you always have the operator that you're acting with come from the left and from the right. That is the natural way in which an operator acts on an operator.

You can think of this thing as your operator is being acted upon as having surrounded by a [INAUDIBLE] and a [INAUDIBLE]. And then when the states transform, one transforms with $U$, one transforms with $U$ dagger. So always the action on an operator is with a $U$ and a $U$ dagger.

So if you ask how does the permutation operator act on $B$, you don't ask generally what's the product of $P$ times $B$. You ask this question. This is the question that may have a nice answer. Then we'll see that there's other ways of doing this. So we want to investigate this operator.

So what I can do is let it act $U_i$, $U_j$. So what do we get? We get $P_{21}B_1$. Now, $P_21$ dagger, we saw that it's Hermitian anyway, so it's just $P_{21}$. And now it acts on $U_i^1$. I'll put the $j$ here, and $U_j^2$. So I let up the $P_{21}$ on that state, and that the moves the $i$'s and the $j$'s.

Now, $B_1$ acts on the first Hilbert space. So now we have $P_{21}$ and we have $BU_j^1$ and tensor $U_i^1$.
2. Now, $P_{21}$ is supposed to put the second state in position one and the first state in position two. So this is $U_1 B U_2$. I could put this thing-- $B U_2$.

And then you see, oh, this term is here. So this is nothing else than $B_2$ acting on the same state of the $U_i U_j$, which means-- I guess I could use this blackboard-- that $P_{21} B_1 P_{21}^\dagger$ is $B_2$.

So it has moved you. The operator used to act on the first particle. Two and one changes the first particle with the second. It moved it into the other one.

Similarly, you could do this also. Would not be a surprise to you that $P_{21} B_2 P_{21}^\dagger$ is equal to $B_1$. And you don't have to do the same argument again. You could multiply this equation by $P_{21}$ from the left and $P_{21}^\dagger$ from the right.

These things become one and one, and the operators remain on the other side and gives you this. So this second equation comes directly from the first. You don't have to go through the arguments.

So what is the use of this thing? You may have a Hamiltonian, and you want to understand what it means to have a symmetric Hamiltonian. And these operators allow you to do that.

So for example, you may have an operator $O_{1,2}$. What is an operator $O_{1,2}$? It's an operator build on things that act on one or act on two. So if you want to imagine it, it could be an $O$ that depends on the operator $A$ acting on the first label, an operator $B$ acting on the second label, an operator $C$ on the first label, an operator $D$ on the second label. Could be a very complicated product of those operators acting on all kinds of labels.

Now suppose you act with $P_{21} O_{1,2} P_{21}^\dagger$. Now, the great advantage of having a $P$ and a $P^\dagger$ acting on a string of operators is that it is the same as having a $P$ and a $P^\dagger$ acting on each one. Remember, if you have like $P$ and $P^\dagger$, and it's a unitary operator on $ABC$, it's the same as $P A P^\dagger$, $P A B P^\dagger$, $P C P^\dagger$. It's like acting on each one.

So when you have this $P_{21} P_{21}^\dagger$ acting on this, it's as if each one of those is surrounded by a $P_{21} P_{21}^\dagger$. So each label one will become a label two, and each label two will become a label one. And therefore, this operation is going to give you $O_{2,1}$ for an arbitrary operator acting on these two labels.

Now, it may happen that the operator is symmetric if $O$ is symmetric. By that, we mean $O_{2,1}$
is equal to $O_{1,2}$. If that happens-- if that happens-- then from this equation you would have
$P_{21} O_{1,2} P_{21} \dagger = O_{1,2}$ itself.

And you could multiply by a $P_{21}$ from the right, giving you $P_{21} O_{1,2} = O_{1,2}$ You're
multiplying by a $P_{21}$ from the right that cancels this $P_{21} \dagger$. $P_{21}$. And there you see that
an operator is symmetric if it commutes with the permutation operator.

So if always symmetric, this is true, and this is true, and then finally, $P_{21}$ with $O_{1,2}$
commutator is 0. Oops. Too low. Let me see. It's a commutator. It's 0. So that's basically how
you manipulate these operators on this Hilbert space.