Announcements

This week we continue to study Canonical Transformations.

- Due to the Columbus day holiday I have made this assignment due on Tuesday Oct. 14 rather than on the Monday. Note however that your next assignment (#6) will be posted on Monday Oct.13 and due on Monday Oct. 20 as usual.

Reading Assignment

- The reading for Canonical Transformations is Goldstein Ch.9 sections 9.1-9.7. (We will not discuss active infinitesimal canonical transformations with the same level of detail that Goldstein does in 9.6, but it is still good reading.)

- The reading on the Hamilton-Jacobi equations and Action-Angle Variables is Goldstein Ch.10 sections 10.1-10.6, and 10.8. We will cover more examples of this material on problem set #6.
Problem Set 5

On this problem set there are 5 problems involving canonical transformations, Poisson brackets, and conserved quantities. In the last problem you will apply the Hamilton-Jacobi method to a problem for which you already know the solution.

1. Canonical Transformations [12 points]

In this problem we get some practice with canonical transformations from \((q,p)\) to \((Q,P)\). We will also look at generating functions \(F(q,p,Q,P,t)\), following the notation in Goldstein for \(F_1(q,Q,t), F_2(q,P,t), F_3(p,Q,t),\) and \(F_4(p,P,t)\).

(a) [2 points] Determine two possible generating functions for \(Q_i = q_i\) and \(P_i = p_i\).

(b) [2 points] Find a generating function \(F_1(q,Q,t)\) for: \(Q = p/t\) and \(P = -q t\).

(c) [4 points] For which parameters \(k, \ell, m, n\) is there a generating function \(F_1(q,Q)\) for: \(Q = q^k p^\ell\) and \(P = q^m p^n\)?

(d) [4 points] For a particle with charge \(q\) and mass \(m\) moving in an electromagnetic field the Hamiltonian is given by

\[
H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi
\]  

(1)

where \(\vec{A} = \vec{A}(\vec{x}, t)\) and \(\phi = \phi(\vec{x}, t)\) are the vector and scalar potentials. Here \(\{x_i, p_j\}\) are canonical coordinates and momenta.

Under a gauge transformation of the electromagnetic field:

\[
\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla}f(\vec{x}, t), \quad \phi \rightarrow \phi' = \phi - \frac{\partial f(\vec{x}, t)}{\partial t},
\]

while \(\vec{p} - q\vec{A}\) is unchanged. Show that this is a canonical transformation for the coordinates and momenta of a charged particle, and determine a generating function \(F_2(\vec{x}, \vec{P}, t)\) for this transformation.

2. Harmonic Oscillator [7 points] (Related to Goldstein Ch.9 #24)

(a) [2 points] For constant \(a\) and canonical variables \(\{q,p\}\), show that the transformation

\[
Q = p + ia q, \quad P = \frac{p - ia q}{2ia}
\]

is canonical by using the theorem that allows you to check this by using Poisson brackets.

(b) [5 points] With a suitable choice for \(a\), obtain a new Hamiltonian for the linear harmonic oscillator problem \(K = K(Q,P)\). Solve the equations of motion with \(K\) to find \(Q(t), P(t)\), and then find \(q(t)\) and \(p(t)\).
3. Poisson Brackets and Conserved Quantities [4 points]

A system of two degrees of freedom is described by the Hamiltonian

\[ H = q_1 p_1 - q_2 p_2 + a q_1^2 + b q_2^2, \]

with constants \( a \) and \( b \). Show that \( u_1 = (p_1 + a q_1)/q_2 \) and \( u_2 = q_1 q_2 \) are constants of the motion.

4. Angular Momentum and the Laplace-Runge-Lenz vector [13 points]

Consider the angular momentum \( \vec{L} = \vec{x} \times \vec{p} \) for canonical variables \( \{x_i, p_j\} \) in 3-dimensions. The components can be written as \( L_i = \epsilon_{ijk} x_j p_k \) with an implicit sum on the repeated indices \( j \) and \( k \). Here \( \epsilon_{ijk} \) is the Levi-Civita tensor

\[ \epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk = 123 \text{ or a cyclic combination of this} \\ -1 & \text{if } ijk = 321 \text{ or a cyclic combination of this} \\ 0 & \text{otherwise} \end{cases} \]

Often \( \epsilon_{ijk} \) is handy when we are considering cross-products: \( \vec{c} = \vec{a} \times \vec{b} \) is equivalent to \( c_i = \epsilon_{ijk} a_j b_k \). Some properties you may find useful are: \( \epsilon_{ijk} = \epsilon_{kji}, \epsilon_{jik} = -\epsilon_{ijk}, \) and \( \sum_k \epsilon_{ijk} \epsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \).

(a) [4 points] As warm up, calculate the Poisson brackets \([x_i, L_j], [p_i, L_j], [L_i, L_j], \) and \([L_i, \vec{L}^2] \).

Now consider two particles attracted to each other by a central potential \( V(r) = -k/r \), where \( r = |\vec{r}| \) is the distance between them. Taking the origin at the CM, the Hamiltonian for this system is \( H = \vec{p}^2/(2\mu) - k/r \) where \( \mu \) is the reduced mass and the \( r_i \) and \( p_j \) are canonical variables. The angular momentum, \( \vec{L} = \vec{r} \times \vec{p} \), is conserved so you may assume that \([L_i, H] = 0 \) (some of you may recall proving this in 8.223).

(b) [7 points] Show that the Laplace-Runge-Lenz vector, \( \vec{A} = \vec{p} \times \vec{L} - \mu k \vec{r}/r, \) is conserved.

Recall that the conservation of \( \vec{L} \) implies that the motion of the particles in this central force take place in a plane that is perpendicular to \( \vec{L} \). The set of \( H, \vec{L}, \vec{A} \) gives 7 constants of motion, but for two particles there are at most 6 constants from integrating the equations of motion. Furthermore, at least one constant must refer to an initial time, and none of \( H, \vec{L}, \vec{A} \) do so. Hence there must be at least two relations between these constants. It is easy to see that \( \vec{L} \cdot \vec{A} = 0 \) provides one relation.

(c) [2 points] Show that the other relation is \( \vec{A}^2 = \mu^2 k^2 + 2\mu H \vec{L}^2 \).

[Read Goldstein section 3.9 to see how \( \vec{A} \) can be used to very easily find the orbital equation \( r = r(\theta) \) for motion in the plane.]
5. **An Exponential Potential** [13 points]

A particle with mass \( m = 1/2 \) is moving along the \( x \)-axis inside a potential \( V(x) = \exp(x) \), so its Hamiltonian is \( H = p^2 + e^x \). You may assume \( p > 0 \).

(a) [6 points] Determine a generating function \( F_2(x, P) \) that yields a new Hamiltonian \( K = P^2 \). (Feel free to check your results with mathematica.)

(b) [3 points] What are the transformation equations \( P = P(x, p) \) and \( Q = Q(x, p) \)?

(c) [4 points] Determine \( x(t) \) and \( p(t) \).

Question [not for points]: How would your analysis change if \( p < 0 \)?

6. **Projectile with Hamilton-Jacobi** [11 points] (Goldstein Ch.10 #17)

Solve the problem of the motion of a point projectile of mass \( m \) in a vertical plane using the Hamilton-Jacobi method. Find both the equation of the trajectory and the dependence of the coordinates on time. Assume that the projectile is fired off at time \( t = 0 \) from the origin with the velocity \( v_0 \), making an angle \( \theta \) with the horizontal.
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