32. Show that for $F(x_1, \ldots, x_n)$, the Legendre transformation is

$$G(s_1, \ldots, s_n) = \sum_{i=1}^{n} x_i s_i - F$$

where

$$s_i = \left( \frac{\partial F}{\partial x_i} \right)_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}$$

and has the property that

$$x_i = \frac{\partial G}{\partial s_i}$$

33. Starting from the Lagrangian for the simple harmonic oscillator,

$$L = \frac{1}{2} m \ddot{x}^2 - \frac{1}{2} k x^2,$$

find the momentum, the Hamiltonian $H(p, q)$ and Hamilton’s equations of motion.

34. Continuing the previous exercise, solve Hamilton’s equations and make a plot of typical trajectories in $(p, q)$ space (referred to as “phase space”.)

35. Give the Hamiltonian in three dimensions for a system with potential $U(r) = -\alpha/r$ using spherical polar coordinates. Write the equations of motion and identify the conserved momenta. Use this result (and a proper reference frame) to reduce the problem to one dynamical variable and associated conjugated momentum (they will be $r$ and $p_r$).

36. OPTIONAL Draw phase space diagrams in $r$ and $p_r$ for the previous problem.

37. Show each of the following fundamental Poisson bracket equations:

$$[q_1, q_2] = 0$$
$$[p_1, p_2] = 0$$
$$[q_1, p_2] = 0$$
$$[p_1, q_1] = 1$$
$$[p_1^2, q_1] = 2p_1$$
$$[p_1, q_1^2] = 2q_1$$
38. Canonical transformation:

(a) Show that the transformation on 2n-dimensional phase space associated with a coordinate transformation on configuration space, namely:

\[ q_i \rightarrow Q_i(q) \]
\[ p_i \rightarrow P_i(q, p) = \sum_j p_j \frac{\partial q_j}{\partial Q_i} \]

is a canonical transformation.

(b) On a 2-dimensional phase space, show that the transformation

\[ q \rightarrow Q = \ln\left(\frac{\sin p}{q}\right) \]
\[ p \rightarrow P = q \cot p \]

is canonical.

(c) Find the generating function \( F_4(p, P) \) of the canonical transformation in (b).

39. As shown in Fig. 1, consider a mass \( m \) attached to a string which in turn is nailed to point \( A \) on a circular spool of radius \( R \). The whole system lies on the horizontal plane, and the spool is fixed so it cannot rotate. As the mass slides without friction, the string remains taut and either winds or unwinds around the spool. \( B \) is the point where the string leaves the spool. Let the total length of the string be \( l \), and let \( s \) denote the free length of the string, that is, the length from \( B \) to the mass. We align the coordinate axes so that the center \( O \) of the spool corresponds to \( x = y = 0 \) and the radius to \( OA \) is in the positive \( y \) direction. Let \( \theta \) denote the angle between \( OA \) and \( OB \).

(a) Express the coordinates \((x, y)\) of the mass in terms of \( s, \theta \) and \( R \). Using the constraint relating \( s \) and \( \theta \) to the total length, find the Lagrangian \( L(s, \dot{s}) \) of the system in terms of the dynamical coordinates \( s \) and its associated velocity \( \dot{s} \).
(b) Find the Hamiltonian $H(s, p)$, write Hamilton’s equations, and confirm that $\frac{p}{s}$ is a constant of the motion. Use this information to find $s(t)$ in terms of its initial value $s_0$ and the total energy $E$.

(c) Consider the following change of coordinates:

$$Q = s^2, \quad P = \frac{p}{\lambda s}$$

Find the value of the constant $\lambda$ so that the transformation is canonical and give $H'(Q, P)$.

40. For a Lagrangian with a velocity dependent potential

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - \lambda y\dot{x},$$

a) Show that the solutions have the form

$$x(t) = -R \cos(\frac{\lambda}{m} t + \phi) + x_0$$

$$y(t) = R \sin(\frac{\lambda}{m} t + \phi) + y_0.$$  

b) Find the conjugate momenta.