1 Last Time

Hamiltonian Mechanics

\[ H(p, q, t) = pq - L(q, \dot{q}, t) \]

Canonical Equations

\[ \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \]

\[ \frac{\partial H}{\partial t} = \frac{dH}{dt} \Rightarrow \frac{\partial H}{\partial t} = 0 \Rightarrow H = \text{const} \]

and often \[ H = E = T + U \]

2 Poisson Brackets

... how to find conserved quantities in any problem.

Let’s say we have some quantity defined as a function of \( p, q, \) and \( t \) (e.g. \( T \) or \( U \) or \( E \) or \( L_z \) or whatever) and we want to know if it is a conserved quantity (constant in time). How can we tell?
### Poisson Brackets

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p} \dot{p}
\]

\[
= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q}
\]

\[
= \frac{\partial f}{\partial t} + [H, f]
\]

where \( [H, f] = \frac{\partial H}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial f}{\partial p} \)

So, if \( f \) is conserved and not an explicit function of time, its Poisson bracket with \( H \) is zero.

\[
[H, f] = \frac{df}{dt} - \frac{\partial f}{\partial t} \quad \text{so if} \quad \frac{\partial f}{\partial t} = 0
\]

\[\text{then} \quad [H, f] = 0 \iff \frac{df}{dt} = 0 \]

Interestingly, for any \( f(t) \), \([H, f] = 0 \) simply because \( \frac{\partial f}{\partial p} = \frac{\partial f}{\partial q} = 0 \), or because \( \frac{df}{dt} - \frac{\partial f}{\partial t} = 0 \). Let’s try this out...

### Example: Gravity, what is \( \frac{dT}{dt} \)?

\( H = T + U = \frac{p^2}{2m} + mgz \)

We will need a bunch of partial derivatives, so let’s compute those first...

\[
\frac{\partial H}{\partial p} = \frac{p}{m}, \quad \frac{\partial H}{\partial z} = mg, \quad \frac{\partial T}{\partial z} = 0, \quad \frac{\partial T}{\partial p} = \frac{p}{m}
\]

\[
[H, T] = \frac{\partial H}{\partial p} \frac{\partial T}{\partial z} - \frac{\partial H}{\partial z} \frac{\partial T}{\partial p} = -gp
\]

\[
= \frac{d}{dt} T
\]
Thus, the rate of change of kinetic energy is \(-gp\). We can confirm this easily with

\[
\frac{dE}{dt} = 0 \Rightarrow \frac{dT}{dt} = -\frac{dU}{dt} = -mg\ddot{z} = -gp
\]

The general definition of the Poisson Bracket for any two functions in an \(N\) degrees of freedom problem is

\[
[f, g] = \sum_{i}^{N} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)
\]

and it has certain properties worth knowing

\[
[f, g] = -[g, f] \quad [f, \alpha] = 0 \quad [f, f] = 0
\]

\[
[f + g, h] = [f, h] + [g, h] \quad \text{(distributive)}
\]

\[
[f \cdot g, h] = f[g, h] + g[f, h] \quad \text{(product rule)}
\]

\[
\Rightarrow [f^2, g] = 2f[f, g]
\]

\[
[f, q_i] = \frac{\partial f}{\partial p_i} \quad [f, p_i] = -\frac{\partial f}{\partial q_i} \quad \text{(see definition)}
\]

\[
\Rightarrow [q_j, q_k] = 0 \quad [p_j, p_k] = 0 \quad [p_j, q_k] = \delta_{jk}
\]

The last and possibly most interesting of these is complicated enough to have a name

**Jacobi’s Identity**

\[
[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0
\]

which I will not prove here.

This brings us to Poisson’s Theorem.
Poisson’s Theorem

\[
\text{if } \frac{df}{dt} = 0 \text{ and } \frac{dg}{dt} = 0 \text{ then } \frac{d}{dt} [f, g] = 0
\]

which in words is: if \( f \) and \( g \) are not explicit functions of time, and they are conserved, their Poisson Bracket is also conserved. Here is the proof:

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} = 0, \quad \frac{dg}{dt} = \frac{\partial g}{\partial t} = 0 \Rightarrow [H, f] = [H, g] = 0
\]

Jacobi \( \Rightarrow \) \( [H, [f, g]] + [f, [g, H]] + [g, [H, f]] = 0 \)

\( \Rightarrow \) \( [H, [f, g]] = 0 \Rightarrow \frac{d}{dt} [f, g] = 0 \)

Let’s try a moderately complicated example of Poisson’s Theorem. Consider a system in which we know that angular momentum \( L_x \) and \( L_y \) are conserved. What can we say about \( L_z \)? Any thing?

\[
\vec{L} = \vec{r} \times \vec{p}
\]

\[
L_x = yp_z - zp_y
\]

\[
L_y = zp_x - xp_z
\]

\[
L_z = xp_y - yp_x
\]

Poisson \( \Rightarrow [L_x, L_y] = \) some conserved quantity

We will need some partial derivatives to compute the PB of \( L_x \) and \( L_y \).

\[
\frac{\partial L_x}{\partial \vec{r}} = \{0, p_z, -p_y\}, \quad \frac{\partial L_x}{\partial \vec{p}} = \{0, -z, y\}
\]

\[
\frac{\partial L_y}{\partial \vec{r}} = \{-p_z, 0, p_x\}, \quad \frac{\partial L_y}{\partial \vec{p}} = \{z, 0, -x\}
\]
\[ [L_x, L_y] = \frac{\partial L_x}{\partial \dot{p}} \cdot \frac{\partial L_y}{\partial \dot{r}} - \frac{\partial L_y}{\partial \dot{r}} \cdot \frac{\partial L_x}{\partial \dot{p}} = y p_x - x p_y = -L_z \]

Thus, if \( L_x \) and \( L_y \) are conserved, so is \( L_z \)! This is for any potential. (\( L_z \) and \( L_y \) must be conserved for any initial conditions, not constrained.) We already know \( \tilde{L} \) is conserved for a central potential, but let me show you how to prove it with Poisson Brackets...

\[
\text{if } [H, L_z] = 0 \Rightarrow L_z = \text{const} \\
[T + U, L_z] = [T, L_z] + [U, L_z]
\]

Let’s do these one at a time...

\[
[T, L_z] = [T, x p_y - y p_x] = [T, x p_y] - [T, y p_x]
\]

\[= ([T, x] p_y + [T, p_y] x) - ([T, y] p_x + [T, p_x] y)\]

It doesn’t look like we are winning, but what I am doing is breaking this down into small enough parts that I can use my identities. For instance,

\[
[T, p_x] = \frac{1}{2m} \left[ p_x^2 + p_y^2 + p_z^2, p_x \right] \\
[p_x^2, p_x] = 2p_x \left[ p_x, p_x \right] = 0
\]

\[\Rightarrow [T, p_i] = 0 \quad \forall i\]

since \( T = \frac{p^2}{2m} \) contains no \( q_i \), i.e. \( \frac{\partial T}{\partial q} = 0 \) and \( [p_j, p_k] = 0 \) (since \( \frac{\partial p}{\partial q} = 0 \)), and

\[
[T, x] = \frac{1}{2m} \left[ p_x^2, x \right] = \frac{p_x}{m} \left[ p_x, x \right] = \frac{p_x}{m}
\]

note that \( [p_y, x] = [p_z, x] = 0 \). Finally,
\[ [T, L_z] = \frac{p_x p_y}{m} - \frac{p_y p_x}{m} = 0 \]

\[ \Rightarrow L_z \text{ conserved for free particle! (and } L_x \text{ and } L_y) \]

Well, I guess we knew that. Let’s do \( U \)...


\[ = [U, p_y] x - [U, p_x] y \]

where I have dropped 2 terms since \([q_j, q_k] = 0\) and \( U(r) \) has no \( p_i \) in it.

\[ [U, p_y] = -\frac{\partial U}{\partial y} = -\frac{\partial U}{\partial r} \frac{\partial r}{\partial y} \]

\[ = -\frac{\partial U}{\partial r} \frac{y}{r} \]

where \( r = \sqrt{x^2 + y^2 + z^2} \Rightarrow \frac{\partial r}{\partial y} = \frac{1}{2r} \frac{2y}{r} = \frac{y}{r} \)

Putting these together to find the PB of \( L_z \) with \( U \),

\[ [U, L_z] = [U, p_y] x - [U, p_x] y \]

\[ = -\frac{\partial U}{\partial r} \left( \frac{y}{r} x - \frac{x}{r} y \right) = 0 \]

That is how Poisson Bracket manipulation works. Break it down until you hit an identity and do your best to never actually compute the derivatives.

For those of you who have taken 8.04, all of this should look VERY familiar. Poisson Brackets are the commutators of classical mechanics, and they work in an analogous manner. For those of you who will take 8.04 soon, remember this, because much of QM hinges on commutators!