Problem 1. (10 points) Boundary conditions for open strings.
Consider two static D2-branes in four dimensional spacetime \((ct, x^1, x^2, x^3)\). The first one is at \(x^3 = 0\). The second one is parallel to the first and is located at \(x^3 = a > 0\). Sketch the branes.
Consider open strings with \(\sigma \in [0, \sigma_1]\) that stretch from the second brane \((\sigma = 0)\) to the first brane \((\sigma = \sigma_1)\).
State the boundary conditions (Free or Dirichlet, with value) for the string coordinates \(X^\mu(t, \sigma_*)\) (list the eight conditions \(- \mu = 0, 1, 2, 3\) and \(\sigma_*=0, \sigma_1\)).

Problem 2. (10 points) Spaces constructed by identifications.
Give a (simple!) fundamental domain \(\mathcal{F}\) and describe the resulting space \(\mathcal{M}\) for each of the following (single) identifications acting on the complex plane \(z = x + iy\):

(a) \(z \sim z + i\).

(b) \(z \sim 2z\).

Problem 3. (15 points) Variation of an action
Consider the Chern-Simons action for three-dimensional electromagnetism:

\[
S = \int dt \int d^2 x (A_0 F_{12} + A_1 F_{20} + A_2 F_{01}).
\]

Recall that the field strength \(F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\). Find the equation of motion resulting from the variation of the gauge field component \(A_0\) (as usual, ignore boundary terms). The equation of motion can be written fully in terms of field strengths.

Problem 4. (10 points) How heavy is a cosmic string?
A nearby relativistic cosmic string of tension \(T_0\) produces a cylindrical gravitational lens in which two images of a single faraway source would be separate by an angle \(\delta = 8\pi GT_0\).

This formula is given in units where \(c\) and \(\hbar\) are set equal to one, the angle \(\delta\) is measured in radians, and \(G \approx 6.7 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2\) is Newton’s constant \((c = 3 \times 10^8 \text{ m/s}, \hbar = 1.06 \times 10^{-34} \text{ kg} \cdot \text{m}^2/\text{s})\).

(a) Complete (1) by adding whatever factors of \(c\) and/or \(\hbar\) are needed.

(b) A string produces the plausible value of \(\delta = 0.5\) arc-seconds (degree = 60 arc-minutes, arc-minute = 60 arc-seconds). What is the linear mass density of such string in kg/m?
Problem 5. (20 points) Angular momentum of a rotating open string.

An open string of length $\ell$ and energy $E$ rotates rigidly with angular velocity $\omega$. Recall that $\omega_{\frac{E}{2}} = c$ and $\ell = \frac{2E}{\pi T_0}$.

(a) Introduce a radial length $r$ along the string and let $dr$ denote a small piece of string a distance $r$ from the center. What is the magnitude $dp$ of the (relativistic) momentum carried by this small piece of string? What is the magnitude $dJ$ of the angular momentum carried by this small piece of string? Both answers should be in terms of $\omega, T_0, r, dr$ and constants.

(b) Use integration to calculate the total angular momentum carried by the rotating string. Give your answer in terms of the energy $E$ of the string and the string tension $T_0$.

Useful integral: $\int_0^1 \frac{x^{2}dx}{\sqrt{1-x^2}} = \frac{\pi}{4}$.

Problem 6. (25 points) Momentum of closed strings.

For a free closed string we have

$$\vec{X}(t, \sigma) = \frac{1}{2} \left( \vec{F}(u) + \vec{G}(v) \right), \quad \text{with} \quad u = ct + \sigma, \ v = ct - \sigma.$$  \hspace{1cm} (1)

(a) Demonstrate that the periodicity condition $\sigma \sim \sigma + \sigma_1$ $(\sigma_1 = E/T_0)$ relates the lack of periodicity of $\vec{F}(u)$ to the lack of periodicity of $\vec{G}(v)$.

(b) We now write

$$\vec{F}(u) = \vec{f}(u) + \vec{\alpha} u, \quad \text{and} \quad \vec{G}(v) = \vec{g}(v) + \vec{\beta} v,$$

where $\vec{f}$ and $\vec{g}$ are strictly periodic functions with period $\sigma_1$ and $\vec{\alpha}$ and $\vec{\beta}$ are constant vectors.

How does the result in (a) relate $\vec{\alpha}$ and $\vec{\beta}$?

Plug back in (1) to find $\vec{X}(t, \sigma)$ in terms of $\vec{f}(u), \vec{g}(v), \vec{\alpha}, ct$, and possibly $\sigma$.

(c) The momentum density (per unit $\sigma$) carried of the string is $\vec{P}^\tau = \frac{T_0}{c^2} \frac{\partial \vec{X}}{\partial t}$. Calculate the total momentum $\vec{p}$ carried by the string in terms of the vector $\vec{\alpha}$ and other constants.

Problem 7. (10 points) You learned that a closed string stretched along a circle and released with zero initial velocity will contract to zero size at some later time. Consider a closed string that is stretched along an ellipse and is released with zero initial velocity. Will it contract to zero size? If yes, why? If not, why not? A complete answer requires a precise justification but, in fact, no calculation.