Lecture 17 - Topics

- Light-cone fields and particles (cont’d.)

Reading: Sections 10.2-10.4

What are we doing now: Preparing grounds to see what arises from the string. How are particles described: Begin with simplest particle/field: the scalar field.

Lagrangian density for a scalar field $\phi(x)$:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \left[ \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} M^2 \phi^2 \right]$$

The first term represents the KE density and the second term represents the PE density.

Note since KE density has same units as PE density:

$$\left[ \frac{1}{2} (\partial_\mu \phi)^2 \right] = \left[ \frac{1}{2} M^2 \phi^2 \right] \Rightarrow [M]$$

$$\mathcal{L} = -\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} M^2 \phi^2$$

$$S = \int d\tilde{x} dt \mathcal{L}$$

$$E = \int H d\tilde{x} = \int d\tilde{x} \left( \frac{1}{2} (\partial_\phi \phi)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} M^2 \phi^2 \right)$$

$$\delta S = \int d\tilde{x} dt (-\eta^{\mu\nu} \partial_\mu (\delta \phi)_\nu - M^2 \phi \delta \phi)$$

$$= \int d\tilde{x} dt \delta \phi (\eta^{\mu\nu} \partial_\mu \partial_\nu \phi - M^2 \phi)$$

$$\frac{\partial^2 \phi}{\partial t^2} - M^2 \phi = 0$$

$$\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi - M^2 \phi = 0$$

This is the equation of motion of scalar field.

Next: Develop notion of scalar particles. How do we recognize them?
Plane Waves

Set scalar field to something that could satisfy equation of motion. Try:

$$\phi = a \exp(-iEt + i\vec{p} \cdot \vec{x})$$

Then:

$$-(iE)^2 + (i\vec{p}) \cdot (i\vec{p}) - M^2 = 0$$

$$E^2 - p^2 = M^2 \Rightarrow -p^2 = M^2 \quad (\text{where } p = p_\mu p^\mu)$$

This looks sort of like a particle in quantum mechanics, but a bit naive. Try:

$$\phi = a \exp(-iEt + i\vec{p} \cdot \vec{x}) + a^* \exp(iEt - i\vec{p} \cdot \vec{x})$$

Can’t anymore think of a particle with momentum $p$ and energy $E$ since get negative $E$. So abandon that interpretation.

Quantum Field Theory: The fields are dynamical variables and operations.

$$\phi(x) = \int \frac{dp}{(2\pi)^D} \exp(ip \cdot x) \phi(p)$$

$$(\phi(x))^* = \int \frac{dp}{(2\pi)^D} \exp(-ip \cdot x) (\phi(p))^* = \int \frac{dp}{(2\pi)^D} \exp(ip \cdot x) (\phi(-p))^*$$

$$(\phi(x))^* = \int \frac{dp}{(2\pi)^D} \exp(ip \cdot x) (\phi(p))$$
\[ [\phi(p)]^* = \phi(-p) \]

If the value of the field for some \((E_p, \vec{p})\)

So geometrically, the reality condition of a point \((E_p, \vec{p})\) in momentum space in the top hyperboloid is equal to the reality condition of the complex conjugate in the bottom hyperboloid.

\[
(\partial^2 - M^2) \int \frac{d^Dp}{(2\pi)^D} \exp(ip \cdot x)\phi(p) = 0
\]

\[
\int \frac{d^Dp}{(2\pi)^D} (-p^2 - M^2)\phi(p)\exp(ipx) = 0
\]

\[
(p^2 + M^2)\phi(p) = 0 \quad \forall p
\]

Say \(p^2 + M^2 \neq 0\) then \(\phi(p) = 0\)

Say \(p^2 + M^2 = 0\) then \(\phi(p)\) is arbitrary.

This is the complete solution. A little simple sounding, but beautiful geometric interpretation. If not on hyperboloid, field vanishes. If on hyperboloid, field arbitrary (subject to reality condition).

\[ \phi(p) \text{ determines } \phi(-p) = (\phi(p))^* \]

1 degree of freedom in the scalar field. (2 real numbers for two points).

**Field Configuration**

\[ \phi_p(t, \vec{x}) = \frac{1}{\sqrt{\sqrt{2E_p}}} \frac{1}{\sqrt{2\pi}} (a(t)e^{i\vec{p} \cdot \vec{x}} + a^*(t)e^{-i\vec{p} \cdot \vec{x}}) \]

\[ V = L_1L_2L_3 \cdots L_d \]

\[ x^i \approx x^i + L^i \]

\[ p_i(x_i + L_i) = p_i x_i + 2\pi n_i \]

\[ p_i L_i = 2\pi n_i \]

\[ S = \int d\vec{x}dt \left( -\frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} M^2 \phi^2 \right) \]
Can evaluate. Can do $x$ integral, but cannot do $t$ integral since $t$ still arbitrary.

\[ E = \int d\vec{x} H \]

\[ S = \int dt \left( \frac{1}{2E_p} \dot{a}^*(t)a(t) - \frac{1}{2} E_p a^*(t)a(t) \right) \]  

(1)

\[ E = \frac{1}{2E_p} \dot{a}^*(t)\dot{a}(t) + \frac{1}{2} E_p a^*(t)a(t) \]

(2)

\[ a(t) = q_1(t) + iq_2(t) \]

Thus:

\[ S = \sum_{i=1}^{2} \int dt \left( \frac{1}{2E_p} \ddot{q}_i^2 - \frac{1}{2} E_p q_i^2 \right) \]

This is a harmonic oscillator.

\[ p_i = \frac{\partial S}{\partial \dot{q}_i} = \frac{\dot{q}_i}{E_p} \]

\[ p_1 + ip_2 = \frac{1}{E_p} (\dot{q}_1 + i\dot{q}_2) = \frac{\dot{a}(t)}{E_p} \]

Equation of motion:

\[ \ddot{q}_i = -E^2_p q_i \]

\[ \ddot{a}(t) = -E^2_p a(t) \]

\[ a(t) = a_p e^{-iE_p t} + a^*_p e^{iE_p t} \]

No reality condition is needed.

\[ E = H = E_p (a^*_p a_p + a^*_p a^-_p) \]

Let $a_p$, $a^-_p$ be destruction operations. Let $a^*_p \rightarrow a^+_p$, $a^-_p \rightarrow a^+_p$ be creation operations.

\[ [a_p, a^+_p] = 1 = [a^-_p, a^+_p] \]

All other commutators = 0.

How do we check this is okay?
\[ [q_i(t), p_j(t)] = i\delta_{ij} \]

\[ E = H = E_p(a^+_p a_p + a^+_p a^-_p) \]

\[
\phi_p(t, \vec{x}) = \frac{1}{\sqrt{\Omega}} \frac{1}{\sqrt{2E_p}} (a(t)e^{i\vec{p}\cdot\vec{x}} + a^*(t)e^{-i\vec{p}\cdot\vec{x}})
\]

\[
= \frac{1}{\sqrt{\Omega}} \frac{1}{\sqrt{2E_p}} (a_p e^{-iE_p t + ipx} + a^-_p e^{iE_p t + ipx} + a^-_p e^{iE_p t - ipx} + a^+_p e^{iE_p t - ipx})
\]

\[
\phi_p(t, \vec{x}) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p}} \frac{1}{\sqrt{2E_p}} a_p e^{-iE_p t + i(\vec{p} \cdot \vec{x})} + a^+_p e^{iE_p t - i(\vec{p} \cdot \vec{x})}
\]

\[ E = H = \sum_{\vec{p}} E_p a^+_p a_p \]

\[ [a_{\vec{p}}, a^+_\vec{q}] = \delta_{\vec{p},\vec{q}} \]

Define a vacuum state \[ |\Omega\rangle \]:

\[ a_{\vec{p}} |\Omega\rangle = 0 \forall \vec{p} \]

\[ E |\Omega\rangle = 0 \]

Create a state \[ a^+_\vec{p} |\Omega\rangle \]

Momentum Operator: \[ \vec{P} = \sum_{\vec{p}} \vec{p} a^+_\vec{p} a_{\vec{p}} \]. Note \[ \vec{P} |\Omega\rangle = 0 \]

\[ \sum_{\vec{q}} E_q a^+_\vec{q} a_{\vec{q}} |\Omega\rangle = \sum_{\vec{q}} E_q a^+_\vec{q} [a_q, a^+_\vec{q}] |\Omega\rangle = E_{\vec{p}} (a^+_\vec{p} |\Omega\rangle) \]

So call \[ a_{\vec{p}} |\Omega\rangle \] a scalar particle of mass \( M \), momentum \( \vec{p} \), and energy \( E_{\vec{p}} = \sqrt{\vec{p}^2 + M^2} \)

Call a 1-particle state \[ a^+_p a^+_p \ldots a^+_p |\Omega\rangle = n \text{ particle state of total energy } E_{p_1} + E_{p_2} + \ldots + E_{p_n} \] and momentum \( \vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_n \)

\[ (E, p^1, p^2, \ldots, p^d) \leftrightarrow (p^+, p^-, p^I) \]

We have labelled the oscillators by the spatial components of the momentum which determine the energy.

Light-cone oscillators:

\[ p^- = \frac{1}{2p^+} (p^+ p^2 + M^2) \]