

Lecture 18 - Topics

- Open Strings

Still for open string:

Heisenberg operators: $X^I(\tau, \sigma), x^-_0, P^\tau J(\tau, \sigma), p^+$

\[
[X^I(\sigma), P^\tau J(\tau, \sigma')] = i\eta^I J(\sigma - \sigma')
\]

\[
[x^-_0, p^+] = -i
\]

\[
\frac{\partial}{\partial \tau} = 2\alpha' p^+ + \frac{\partial}{\partial X^+} \Leftrightarrow \frac{2\alpha' p^+ p^-}{\text{Hamiltonian, } H} = H
\]

\[
p^- = \int d\sigma (p^-) = \frac{1}{2\pi \alpha'} \frac{\partial X^-}{\partial \tau} (\frac{\partial}{\partial \tau})
\]

$H = 2\alpha' p^+ p^- = L^+_0$ from analysis of classical string

Are we sure $H = 2\alpha' p^+ p^-$? After all, $p^-$ is the product of lots of operators, which can be ill-defined. Must be careful in our quantum case.

\[
\ddot{X}^I - X^{''I} = 0
\]

\[
X^I(\tau, \sigma) = x^I_0 + \sqrt{2\alpha'}\alpha^I_0 \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha^I_n \cos(n\sigma) e^{-in\tau}
\]

\[
P^{\tau J} = \frac{1}{2\pi \alpha'} \frac{\partial x^J}{\partial \tau}
\]

\[
(\dddot{X}^I + X^{''I})(\tau, \sigma) = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha^I_n e^{(-in(\tau + \sigma))} \quad \sigma \in [0, \pi]
\] (1)
\[ (\dot{X}^I - X'^I)(\tau, \sigma) = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha^I_n e^{-i n (\tau - \sigma)} \] \quad \sigma \in [0, \pi] \tag{2} 

This is an important computation. Later, we will do this for closed strings too, and we’ll see very similar (though not same).

Best way to select Fourier modes is in \([0, 2\pi]\) but \(\sigma \in [0, \pi]\). \(\sigma \to -\sigma\)

\[ (\dot{X}^I - X'^I)(\tau, -\sigma) = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha^I_n e^{-i n (\tau + \sigma)} \] \tag{2'}

This makes sense when \(\sigma \in [-\pi, 0]\).

\[ A^I(\tau, \sigma) = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha^I_n e^{-i n (\tau + \sigma)} \quad \sigma \in [-\pi, \pi] \]

\[ = \begin{cases} 
(\dot{X}^I + X'^I)(\tau, \sigma) & \sigma \in [0, \pi] \\
(\dot{X}^I - X'^I) & \sigma \in [-\pi, 0] 
\end{cases} \]

Now have \(\sigma\) defined over \([-\pi, \pi]\).

\[ [X^I(\tau, \sigma), \dot{X}^I(\tau, \sigma')] = 2\pi \alpha' i \eta^{IJ} \delta(\sigma - \sigma') \]
\[ [\dot{X}^I(\tau, \sigma), \dot{X}^J(\tau, \sigma')] = 0 \]

\[ [X'^I(\tau, \sigma), X'^J(\tau, \sigma')] = 0 \] \(X^I\)'s commute at different \(\sigma\)'s so can then differentiate.

\[ \frac{d}{d\sigma}(\dot{X}^I - X'^I)(\tau, \sigma) = \pm 4\pi \alpha' i \eta^{IJ} \frac{d}{d\sigma}(\sigma - \sigma') \]
\[ [A^I(\tau, \sigma), A^J(\tau, \sigma')] = 2\alpha I \sum_{m', n'} e^{-im'(\tau + \sigma)} e^{-in'(\tau + \sigma')} [\alpha_{m'}^I, \alpha_n^J] \]

\[
= \begin{cases} 
4\pi \alpha' \eta^{IJ} \frac{d}{d\sigma} \delta(\sigma - \sigma') & \sigma, \sigma' \in [0, \pi] \\
4\pi \alpha' \eta^{IJ} \frac{d^2}{d\sigma^2} \delta(\sigma - \sigma') = 0 & \sigma \in [0, \pi], \sigma' \in [-\pi, 0] \\
-4\pi \alpha' \eta^{IJ} \frac{d}{d(\sigma - \sigma')} \delta(\sigma' - \sigma) = 4\pi \alpha' \eta^{IJ} \frac{d}{d(\sigma - \sigma')} \delta(\sigma - \sigma') & \sigma, \sigma' \in [-\pi, 0] 
\end{cases}
\]

\[
\sum_{m', n'} e^{-im'(\tau + \sigma)} e^{-in'(\tau + \sigma')} [\alpha_{m'}^I, \alpha_n^J] = 2\pi i \eta^{IJ} \frac{d}{d\sigma} \delta(\sigma - \sigma') \quad \sigma, \sigma' \in [-\pi, \pi]
\]

Apply the following integral operations:

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma e^{im\sigma} \cdot \frac{1}{2\pi} \int_{-\sigma}^{\sigma} d\sigma' e^{in\sigma}
\]

Divide by \(e^{-i(m+n)\tau}\) on both sides:

\[
[\alpha_{m}^I, \alpha_n^J] = -m \eta^{IJ} \delta_{m+n,0} e^{i(m+n)\tau}
\]

\[
[\alpha_{m}^I, \alpha_n^J] = m \delta_{m+n,0} \eta^{IJ}
\]

Commutation relation proved in book:

\[
[x_0^I, p^J] = i\eta^{IJ}
\]

Note:

\[
\alpha_0^I = \sqrt{2\alpha} p^I
\]

\[
[\alpha_{m}^I, \alpha_n^J] = m \eta^{IJ} \delta_{m,n}
\]

\[
\alpha_\mu = \alpha_n^\mu \sqrt{n} \quad n > 0
\]
\[ \alpha_{m-n}^n = \alpha_{n}^n \sqrt{n} = (\alpha_{+n}^n)^+ \quad n < 0 \]

Opposite signs for \( m \) and \( n \)

\[
[a_{m}^l, a_{n}^l] = 0 \\
[a_{m}^{l+}, a_{n}^{l+}] = 0
\]

\( m > 0, n > 0 \):

\[
[a_{m}^l \sqrt{m}, a_{n}^l \sqrt{n}] = mn^{l+j}\delta_{m,n} \\
[a_{m}^{l+} a_{n}^{l+}] = n^{l+j}\delta_{m,n}
\]

\[
\sqrt{2\alpha^l\alpha_n^-} = \frac{1}{p^+}L_n^+ \quad n \neq 0, \quad 2p^+p^- = \frac{1}{\alpha^l}L_0^+ \\
L_n^+ = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{n-p}^l \alpha_p^l
\]

Don’t have to worry if \( n \neq 0 \). Might have to worry if \( n = 0 \).

But what we want is: \( H = L_0^+ = 2\alpha^l p^+p^- \). \( L_0^+ = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{-p}^l \alpha_p^l \) but \( \alpha \)'s don’t commute so don’t know if this is right.

\[
M^2 = -p^2 = 2p^+p^- - p^l p^l = \frac{1}{\alpha^l}L_0^+ - p^l p^l
\]

\[
L_0^+ = \frac{1}{2} \alpha_0^l \alpha_0^l + \frac{1}{2} \sum_{p=1}^{\infty} (\alpha_{-p}^l \alpha_p^l + \alpha_p^l \alpha_{-p}^l) \\
= \alpha^l p^l + \sum_{p=1}^{\infty} \alpha_{-p}^l \alpha_p^l + \frac{1}{2} (D - 2) \sum_{p=1}^{\infty} p
\]

Note \( \alpha_{p>0} \) is destruction operation convention. \( \alpha_{p<0} \) is creation operation convention.
In classical theory, had

\[ M^2 = \frac{1}{\alpha^2} \left( \sum_{p=1}^{\infty} p a_p I^+ a_p I^- + \frac{1}{2} (D - 2) \sum_{p=1}^{\infty} p \right) \]

Showed all states of string had mass \( \geq 0 \). Couldn’t get anything interesting without mass.

Would be great here if \( \frac{1}{2} (D - 2) \sum_{p=1}^{\infty} p = -1 \). Then:

\[ M^2 = \frac{1}{\alpha^2} \left( \sum_{n=1}^{\infty} n a_n I^+ a_n I^- - 1 \right) \]

Now want oscillation states without mass

\[ \sum_{p=1}^{\infty} p = 1 + 2 + 3 + 4 + \ldots = -\frac{1}{12} \]

Crazy, huh? Not true in general, of course, but almost true in one sense. Since we want:

\[ \frac{1}{2} (D - 2) \sum_{p=1}^{\infty} p = -1 \]

\[ \frac{1}{2} (D - 2) \left( -\frac{1}{12} \right) = -1 \Rightarrow D = 26(\text{dimension of string}) \]

Now how is \( \sum_{p=1}^{\infty} p = -\frac{1}{12} \)?!

Recall Riemann Zeta Function:
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]

\[ \zeta(s = -1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n \]

\( \zeta(s) \) well-defined and convergent for \( s \geq 2 \). Doesn’t converge for \( s = 1 \) (pole). \( \zeta \) defined on complex plane.

The beauty of analytic functions: If you know it is defined in a very small finite region, you know it everywhere by the Cauchy-Riemann.

\[ 2p^+p^- = \frac{1}{\alpha^t} (L^+_0 + a) \quad a = \text{constant} \]

Define for once and for all:

\[ L^\pm_0 = \frac{1}{2} \alpha^0_0 \alpha^0_0 + \sum_{p=1}^{\infty} \alpha^p_{-p} \alpha^p_p \]

\[ [M^{-I}(a,D), M^{-I}(a,D)] = 0 \]

Set standards of messy computation. All books omit at least some details.

\[ M^{-J} \approx \alpha^-_n \alpha^J_m \approx [L^+_n, L^+_m] = (m - n) L^+_m n + \text{dim. of spacetime} \]

So need to find algebra of Viroso operators.