Lecture 6 - Topics

• The relativistic point particle: Action, reparametrizations, and equations of motion

Reading: Zwiebach, Chapter 5

Continued from last time.

\[
\frac{\partial P^t}{\partial t} + \frac{\partial P^x}{\partial x} = 0
\]

\( P^t = \mu_0 \partial y/\partial t \)

\( P^x = -T_0 \partial y/\partial x \)

Similar to \( \partial \mu J^\mu = 0, \partial \rho/\partial t + \nabla \cdot \vec{J} = 0, Q = \int dx \rho \)

Free BC (Neumann BC):

\( P^x(t, x_\ast) = 0 \)

\( P_y = \int_0^a \mu_0 dx (\partial y/\partial t) = \int_0^a dx P^t \)

\( \partial P_y/\partial t = \int_0^a dx \partial P^t / \partial t = -\int_0^a dx \partial P^x / \partial x = -[P^x(t, x = a) - P^x(t, x = 0)] \)

Conservation of momentum?

**Free Relativistic Particle**

Non-relativistic Action:

\[
S = \int dt \left( \frac{1}{2} mv^2 \right)
\]

Calculation: \( dv/dt = 0 \)

Relativistic Particles:
Everyone should agree on action. It’s a Lorentz invar.

\[-ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu\]

\[ds = cdt \sqrt{1 - v^2/c^2} = c d\tau\]

\[s = -mc^2 \int_P \frac{ds}{c} = -mc \int_P ds\]

So: \(s = -mc^2 \int_{t_i}^{t_f} dt \sqrt{1 - v^2/c^2}\)

Check:

Lagrangian:

\[L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}\]

\[= -mc^2 \left(1 - \frac{1}{2} \frac{v^2}{c^2} - \ldots\right)\quad \text{Taylor Expansion}\]

\[= -mc^2 + \frac{1}{2} \underbrace{mc^2}_{\text{rest energy}} + \frac{1}{2} \underbrace{mv^2}_{\text{kinetic energy}}\]

Momentum:

\[\vec{p} = \frac{\partial L}{\partial \vec{v}}\]

\[= -mc^2 \cdot \underbrace{\frac{1 - \frac{2\vec{v} \cdot \vec{v}}{c^2}}{2}}_{\sqrt{1 - \frac{v^2}{c^2}}}\]

\[= \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}}\]

Hamiltonian:

\[H = \vec{p} \cdot \vec{v} - L = \ldots = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}\]

Parameterization

Have parameterization \(x^\mu(\tau)\) (the \(x^\mu\)’s are functions of \(\tau\))

\[ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu\]
\[ ds = \sqrt{-\eta_{\mu\nu} \left( \frac{dx^\mu}{d\tau} \right) \left( \frac{dx^\nu}{d\tau} \right) d\tau} \]

\[ s = -mc \int_{t_i}^{t_f} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau \]

\[ \tau'(\tau): \]

\[ \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'} \frac{d\tau'}{d\tau} \]

\[ s = -mc \int_{t_i}^{t_f} \sqrt{-\eta_{\mu\nu} \left( \frac{dx^\mu}{d\tau'} \frac{dx^\nu}{d\tau'} \right) \frac{d\tau'}{d\tau}} d\tau \]

\[ = -mc \int_{t_i}^{t_f} \sqrt{-\eta_{\mu\nu} \left( \frac{dx^\mu}{d\tau'} \frac{dx^\nu}{d\tau'} \right) d\tau'} \]

So using a different parameter, \( \tau' \) (instead of \( \tau \)) gets same action \( s \). \( s \) is reparameterization-invariant.

Quick calculation to find equation of motion from \( s = -mc \int \sqrt{1 - \frac{v^2}{c^2}} dt \). Should get derivative of rel. momentum with respect to time = 0.

\[ S = -mc \int dS \]

\[ \delta S = -mc \int \delta(dS) \]

\[ dS^2 = -\eta_{\mu\nu} dx^\mu dx^\nu \]

\[ (dS)^2 = -\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (d\tau)^2 \]

\[ 2(dS) \cdot \delta(dS) = -2\eta_{\mu\nu} \delta \left( \frac{dx^\mu}{d\tau} \right) \frac{dx^\nu}{d\tau} (d\tau)^2 \]
\[
\delta (dS) = - \eta_{\mu \nu} \frac{d}{d\tau} (\delta x^\mu) \frac{dx^\nu}{ds} d\tau
\]

Must vary with \(dx^\mu/d\tau\) and \(dx^\nu/d\tau\), but since \(\eta_{\mu \nu}\) is symmetric sufficient to vary just \(dx^\mu/d\tau\) and multiply by 2.

\[
\delta (dS) = - \frac{d}{d\tau} (\delta x^\mu) \frac{dx^\mu}{ds} d\tau
\]

\[
\delta S = \int_{\tau_i}^{\tau_f} \frac{d}{d\tau} (\delta x^\mu) \left( mc \frac{dx^\mu}{ds} \right) d\tau
\]

\[
= \int_{\tau_i}^{\tau_f} \left[ \frac{d}{d\tau} (\delta x^\mu P_\mu) - \delta x^\mu \frac{dP_\mu}{d\tau} \right] d\tau
\]

\[
\delta x^\mu (\tau_i) = \delta x^\mu (\tau_f) = 0
\]

\[
dS = - \int_{\tau_i}^{\tau_f} \left( \delta x^\mu (\tau) \frac{dP_\mu}{d\tau} \right) d\tau
\]

Equation of Motion:

\[
\frac{dP_\mu}{d\tau} = 0
\]

This means that \(P_\mu\) constant on world-line. Constant as a function of any parameter!

\[
\frac{dP_\mu}{dt} \bigg|_0 \frac{dP_\mu}{d\tau} \cdot \frac{dt}{d\tau} \neq 0
\]

Therefore: \(\frac{d}{d\tau} \left( \frac{dx^\mu}{ds} \right) = 0\), \(\frac{d^2}{d\tau^2} (dx^\mu) = 0\) (if \(\tau = s\). Okay because \(\tau\) is arbitrary.)

But can’t assign \(s = \tau\): \(d^2 x^\mu/d\tau^2 \neq 0\).

\[
\frac{d}{ds} \left( \frac{dx^\mu}{d\tau} \right) \neq 0
\]
Coupling to Electromagnetism

Lorentz Force Equation:

\[
\frac{dP_\mu}{dS} = \frac{q}{c} \cdot F_{\mu\nu} \frac{dx^\nu}{ds}
\]

\[
\frac{dP_\mu}{d\tau} = \frac{q}{c} \cdot F_{\mu\nu} \frac{dx^\nu}{d\tau}
\]

\[
S = -mc \int_P dS + \frac{q}{c} \int_P A_\mu(x(\tau)) \frac{dx^\mu}{d\tau} d\tau
\]

A: Nevitz-Schwartz Tensor