Lecture 9 - Topics

• Change of variable

1. Change of Variables, 1 Variable

\[
\int f(x)dx = \int \tilde{f}(u) \frac{dx(u)}{du} du
\]

\[
f(x)dx = \tilde{f}(u) \frac{dx}{du} du
\]

\[u = u(x) \text{. Assume invertible function } x(u)\]

\[\tilde{f}(u) = f(x(u))\]

2 Variables of Integration

\[
\int f(\xi^1, \xi^2) d\xi^1 d\xi^2
\]

Let \(M_{ij} = \frac{\partial \xi^i}{\partial \xi^j}\)

\[
dv^i_1 = \frac{\partial \xi^i}{\partial \xi^1} d\xi^1
\]

\[
dv^i_2 = \frac{\partial \xi^i}{\partial \xi^2} d\xi^2
\]

\[
f(\xi^1, \xi^2) d\xi^1 d\xi^2 = f(\xi^1, \xi^2) dA
\]

\[
dA = |d\tilde{v}_1||d\tilde{v}_2| \sin \theta
\]

\[
= \sqrt{|dv_1|dv_2| - (dv_1 \cdot dv_2)}
\]

\[
= \sqrt{(\frac{\partial \xi^1}{\partial \xi^1} \frac{\partial \xi^1}{\partial \xi^1})(\frac{\partial \xi^2}{\partial \xi^2} \frac{\partial \xi^2}{\partial \xi^2}) - (\frac{\partial \xi^1}{\partial \xi^1} \frac{\partial \xi^1}{\partial \xi^1})^2} d\xi^1 d\xi^2
\]
\[ dA = \sqrt{(M_{i1} \cdot M_{i1})(M_{j2} \cdot M_{j2}) - M_{i1} M_{i2} M_{j1} M_{j2} d\tilde{\xi}^1 d\tilde{\xi}^2} \]

\[ M_{i1} = (M^T)_{i1} \]

\[ dA = \sqrt{(M^T M)_{11} (M^T M)_{22} - (M^T M)^2_{12} d\tilde{\xi}_1 d\tilde{\xi}_2} = \sqrt{\det(M^T M)} d\tilde{\xi}_1 d\tilde{\xi}_2 \]

\[ \det(M^T M) = \det(M^T) \det(M) \]

\[ dA = |\det(M)| d\tilde{\xi}_1 d\tilde{\xi}_2 \]

So:

\[ f(\tilde{\xi}_1, \tilde{\xi}_2) d\tilde{\xi}_1 d\tilde{\xi}_2 = \tilde{f}(\tilde{\xi}_1, \tilde{\xi}_2) |\det(\frac{\partial \tilde{\xi}_i}{\partial \xi_j})| d\xi_1 d\xi_2 \]

The goal is to verify: \( A = \int d\xi_1 d\xi_2 \sqrt{g} \) where \( g = \det(g_{ij}) \) is reparam. invariant.

\[ g_{ij} = \tilde{g}_{pq} \tilde{M}_{pi} \tilde{M}_{pj} \]

\[ = (\tilde{M}^T)_{ip} \tilde{g}_{pq} \tilde{M}_{pj} \]

\[ = (\tilde{M}^T \tilde{g} \tilde{M})_{ij} \]

\[ g = \det(g_{ij}) = \det(\tilde{M}^T) \tilde{g} \det(\tilde{M}) = \tilde{g} |\det(\tilde{M})|^2 \]

\[ \det(\tilde{M}^T) = \det(\tilde{M}) \]

\[ A = \int d\xi_1 d\xi_2 \sqrt{\tilde{g}} = \int d\tilde{\xi}_1 d\tilde{\xi}_2 \det(u) \sqrt{\tilde{g}} \det(\tilde{M}) \]

\[ (\tilde{M} \tilde{M})_{ij} = M_{ik} \tilde{M}_{kj} = \frac{\partial \xi_i}{\partial \xi^k} \frac{\partial \xi^k}{\partial \tilde{\xi}_j} = \frac{\partial \xi_i}{\partial \tilde{\xi}_j} \]

If \( i \neq j \), this equals 0. If \( i = j \), this equals 1. Therefore, we have \( \delta^i_j \).
\[ \text{det}(M) = \text{det}(\tilde{M}) \]

\[ A = \int d\tilde{\xi}^1 d\tilde{\xi}^2 \sqrt{\tilde{g}} \]

Goal: Write area functional for spacetime surface. Just did this for a surface in Euclidean space. Now do for a surface in Minkowski space (so there's a negative sign instead of all positive signs).

Change of notation: \((\xi^1, \xi^2) \rightarrow (\tau, \sigma)\) where \(\tau\) is “like time” and \(\sigma\) is “like time”.

Target space:

\[ x^{\mu} = (x^0, x^1, \ldots, x^d) \]

\( D = d + 1 = \text{space time dimension. } d = \text{spatial dimension.} \)

Mapping:

\[ x^{\mu}(\tau, \sigma) = X^{\mu}(\tau, \sigma) \]

\(X^{\mu}\) sometimes called “string coordinates”. \(\sigma\) has a finite range. For a closed string, periodic. \(\tau\) can have an infinite range.

Area constructed from: \(dv^1_{\mu} = \frac{dX^\mu}{d\tau} \ d\tau, \ dv^2_{\mu} = \frac{dX^\mu}{d\sigma} \ d\sigma. \)

By analogy: \(dA = \sqrt{(dv_1 \cdot dv_1)(dv_2 \cdot dv_2) - (dv_1 \cdot dv_2)^2} \)?

The problem is that the number under the square root is less than 0, and we don’t want an imaginary \(dA!\)

Static String:

\[ X^0(\tau, \sigma) = c\tau \]

\[ X^i(\tau, \sigma) = f^i(\sigma) \]

\(dv_1^\mu\) has only \(\mu = 0\) component.

\(dv_2^\mu\) has only \(\mu \neq 0\) components.

\[ dv_1 \cdot dv_1 < 0 \]

\[ dv_2 \cdot dv_2 > 0 \]

\[ dv_1 \cdot dv_2 = 0 \]

Therefore:
\[ dA = \sqrt{<0} \]

So instead:

\[
dA = d\tau d\sigma \sqrt{ \left( \frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma} \right)^2 - \left( \frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \tau} \right) \left( \frac{\partial X}{\partial \sigma} \cdot \frac{\partial X}{\partial \sigma} \right) }
\]

\[ = d\tau d\sigma \sqrt{ \left( \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\mu}{\partial \sigma} \right)^2 - \left( \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\mu}{\partial \tau} \right) \left( \frac{\partial X^\mu}{\partial \sigma} \frac{\partial X^\mu}{\partial \sigma} \right) } \left( \cdots \right) \]

Consider worldline \( x^\mu(\tau) \): \( \frac{dx^\mu}{d\tau} \) is timelike. Particle moves slower than light.

Consider point \( P \) on worldsheet of a string. Worldsheet described by \( \tau \) and \( \sigma \) so:

\[ X_p = X^\nu(\tau_p, \sigma_p) \]

If all points \( P \) on string \( \exists \) a point \( P' \) on string at time \( \Delta t \). \( u_{\nu' \rho} \) is timelike then string moving slower than light.

The worldsheet is the area swept out by the string over time.
1. \( \forall P \exists \) spacelike tangent
2. Just saw \( \forall P \exists \) timelike tangent too.

Will use 1 and 2 to show \( dA = \sqrt{> 0} \).

Consider tangent vectors at \( P \) spanned by \( \frac{\partial X^\mu}{\partial \tau}(P), \frac{\partial X^\mu}{\partial \sigma}(P) \).

Consider 1 parameter family of vectors.

\[
v^\mu(\lambda) = \frac{\partial X^\mu}{\partial \tau} + \lambda \frac{\partial X^\mu}{\partial \sigma}
\]

Linear combination of \( \frac{\partial X^\mu}{\partial \tau} \) and \( \frac{\partial X^\mu}{\partial \sigma} \) with coefficients 1\&\( \lambda \). Most agreed would have 2 arbitrary coefficients, but here only care about direction.

\[
v^2(\lambda) = \lambda^2 \left( \frac{\partial X^\mu}{\partial \sigma} \right)^2 + \partial x \left( \frac{\partial X^\mu}{\partial \sigma} \cdot \frac{\partial X^\mu}{\partial \tau} \right) + \left( \frac{\partial X^\mu}{\partial \tau} \right)^2
\]

Get quadratic equation for \( \lambda \)

\[
\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
If $b^2 - 4ac \leq 0$ will get complex (not real) roots. So $b^2 - 4ac > 0$. 