Consider a free electron at \( y = 0 \) in the presence of a passing electromagnetic wave of frequency \( \omega \) and electric-field amplitude \( E_0 \):

\[
\lambda = \frac{2\pi c}{\omega}
\]

With \( E_z(y,t) = E_0 \sin(\omega t - ky) \). The equation of motion for the electron is:

\[
m \ddot{z} = -eE_0 \sin \omega t.
\]

The instantaneous acceleration is:

\[
A = \ddot{z} = -\frac{eE_0}{m} \sin \omega t.
\]

The instantaneous power radiated by an accelerating charge (derived in 8.03 - hopefully) is given by

\[
P = \frac{2}{3} \frac{e^2 A^2}{c^3}.
\]

Evaluate this expression with the acceleration of the oscillating electron:
\[ P = \frac{2}{3} \frac{e^4 E_0^2}{m^2 c^3} \sin^2 \omega t. \]

As defined in lecture, the scattering cross section \( \sigma \) is given by

\[ \sigma = \frac{\text{Scattered Power}}{\text{Incident Flux}}. \]

However, the incident energy flux in a plane electromagnetic wave is easily obtained from the Poynting vector (from 8.03):

\[ \text{Flux} = \frac{c}{4\pi} E_0^2 \sin^2 \omega t. \]

Thus, we find:

\[ \sigma_{\text{Thomson}} = \frac{8\pi}{3} \frac{e^4}{m^2 c^4} = \frac{8\pi}{3} \left( \frac{e^2}{mc^2} \right)^2. \]

\[ \frac{e^2}{mc^2} = \text{classical electron radius} \ (2.8 \times 10^{-13} \text{ cm}) \]

**Rayleigh Scattering**

Consider a simple classical model of an atom:

- **nucleus** \( \rightarrow \) \( \uparrow \) **electron**
- Spring (with spring constant \( k \))
The equation of motion for the electron is the same as that given above for the free electron, with the addition of a "restoring force" term, \(-kz\):

\[
m \ddot{z} = -kz - eE_0 \sin wt
\]

Try a solution of the form \(z = A \sin wt\), where \(A\) is a constant to be determined.

\[-mw^2 A \sin wt = -kA \sin wt - eE_0 \sin wt,
\]

which yields

\[
A = \frac{(eE_0/m) \sin wt}{(w^2 - k/m)}.
\]

The instantaneous acceleration \((\ddot{z} = -w^2 A \sin wt)\) is:

\[
\ddot{z} = \frac{\ddot{z} = -w^2 (eE_0/m) \sin wt}{(w^2 - k/m)}.
\]

The cross section will be given by:

\[
\sigma = \frac{\frac{2}{3} e^2 a^2}{c^3} \frac{s}{4\pi} E_0^2 \sin^2 wt,
\]

as in the previous case.
\[ \sigma = \frac{8\pi \epsilon_0^4 \omega^4}{3 m^2 c^4 (\omega^2 - k/m)^2} \]

\[ \sigma_{\text{Rayleigh}} = \sigma_{\text{Thomson}} \frac{\omega^4}{(\omega^2 - k/m)^2} \]

In this expression, the quantity \( k/m \) is the square of the natural frequency of the electron on a spring system, which we call \( \omega_0 \). When the incident radiation has a frequency close to the natural frequency of the system, the cross section becomes very large! For the case where the incident radiation has a frequency well below that which is required to excite the atom, i.e., \( \omega \ll \omega_0 \), the cross section becomes:

\[ \sigma_{\text{Rayleigh}} \approx \sigma_{\text{Thomson}} \left(\frac{\omega}{\omega_0}\right)^4 = \sigma_{\text{Thomson}} \left(\frac{\lambda_0}{\lambda}\right)^4 \]