8.286 Lecture 12  
October 22, 2013  

Non-Euclidean Spaces: Open Universes and the Spacetime Metric

Polar Coordinates:  
\[
x = R \sin \theta \cos \phi \\
y = R \sin \theta \sin \phi \\
z = R \cos \theta ,
\]

Summary of Lecture 11:  
Surface of a Sphere  
\[x^2 + y^2 + z^2 = R^2.\]

Varying \(\theta\):  
\[ds = R \, d\theta\]
Varying $\phi$:

$$ds = R \sin \theta \, d\phi$$

Varying $\theta$ and $\phi$

Varying $\theta$:  
$$ds = R \, d\theta$$

Varying $\phi$:  
$$ds = R \sin \theta \, d\phi$$

$$ds^2 = R^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right)$$

Review of Lecture 11: A Closed Three-Dimensional Space

$$x^2 + y^2 + z^2 + w^2 = R^2$$

$$x = R \sin \psi \sin \theta \cos \phi$$
$$y = R \sin \psi \sin \theta \sin \phi$$
$$z = R \sin \psi \cos \theta$$
$$w = R \cos \psi$$

$$ds = R \, d\psi$$

Metric for the Closed 3D Space

Varying $\psi$:  
$$ds = R \, d\psi$$

Varying $\theta$ or $\phi$:  
$$ds^2 = R^2 \sin^2 \psi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right)$$

If the variations are orthogonal to each other, then

$$ds^2 = R^2 \left[ d\psi^2 + \sin^2 \psi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right]$$
Proof of Orthogonality of Variations

Let \(d\vec{R}_\psi\) = displacement of point when \(\psi\) is changed to \(\psi + d\psi\).
Let \(d\vec{R}_\theta\) = displacement of point when \(\theta\) is changed to \(\theta + d\theta\).

\[\begin{align*}
\star \quad d\vec{R}_\theta & \text{ has no } w\text{-component } \implies d\vec{R}_\psi \cdot d\vec{R}_\theta = d\vec{R}_\psi \cdot d\vec{R}_\theta^{(3)},
\end{align*}\]
where (3) denotes the projection into the \(x-y-z\) subspace.

\[\begin{align*}
\star \quad d\vec{R}_\psi^{(3)} & \text{ is radial; } d\vec{R}_\theta^{(3)} \text{ is tangential } \\
\implies d\vec{R}_\psi^{(3)} \cdot d\vec{R}_\theta^{(3)} & = 0
\end{align*}\]

Review of Lecture 11:
Implications of General Relativity

\[ds^2 = R^2 \left[ d\psi^2 + \sin^2 \psi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right],\] where \(R\) is radius of curvature.

\[\star \quad \text{According to GR, matter causes space to curve.}\]

\[\star \quad R \text{ cannot be arbitrary. Instead, } R^2(t) = \frac{a^2(t)}{k}.\]

\[\star \quad \text{Finally, } \]
\[ds^2 = a^2(t) \left\{ \frac{dr^2}{1-kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right\},\]

where \(r = \frac{\sin \psi}{\sqrt{k}}\). Called the Robertson-Walker metric.
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