8.286 Lecture 13
October 24, 2013

NON-EUCLIDEAN SPACES: SPACETIME METRIC AND THE GEODESIC EQUATION

Summary, Cont: Why is This the Open Universe Metric?

Open Universe:

\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 + \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} , \]

where \( \kappa = -k > 0. \)

Requirements: Isotropy and Homogeneity

Isotropy about the origin is obvious: \( \theta \) and \( \phi \) appear as

\[ a^2(t)r^2(d\theta^2 + \sin^2 \theta d\phi^2) , \]

exactly as on a sphere of radius \( a(t)r. \)

Summary, Cont: Why is This Homogeneous?

Open Universe:

\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} , \]

with \( k < 0. \)

For closed universe (\( k > 0 \)), show homogeneity explicitly by showing that any point \( (r_0, \theta_0, \phi_0) \) is equivalent to the origin:

Construct a map \( (r, \theta, \phi) \rightarrow (r', \theta', \phi') \) which preserves metric, and maps \( (r_0, \theta_0, \phi_0) \) to the origin, \( (r' = 0, \theta', \phi') \).

Construct map in three steps:

\( (r, \theta, \phi) \rightarrow (x, y, z, w) \quad \rightarrow (x', y', z', w') \rightarrow (r', \theta', \phi') . \)

The same map works for \( k < 0 \), showing that any point can be mapped to the origin by a metric-preserving mapping. (We will not show it.)

Closed Universe:

\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} , \]

where \( k > 0. \)

Open Universe: **Same thing, but with \( k < 0! \)**

Name: **Robertson-Walker metric.**
We will not show it, but any 3D homogeneous isotropic space can be described by the Robertson-Walker metric, for $k$ positive, negative, or zero (flat universe).

For $k > 0$, the universe is finite. For $k \leq 0$, the universe is infinite.

The Gauss–Bolyai–Lobachevsky geometry is the 2-dimensional open universe.

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**Summary, Cont: From Space to Spacetime**

In special relativity,

\[ s_{AB}^2 = (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2(t_A - t_B)^2. \]

$s_{AB}^2$ is Lorentz-invariant — it has the same value for all inertial reference frames.

Meaning of $s_{AB}^2$:

If positive, it is the distance between the two events in the frame in which they are simultaneous. (Spacelike.)

If negative, it is the time interval between the two events in the frame in which they occur at the same place. (Timelike.)

If zero, it implies that a light pulse could travel from $A$ to $B$ (or from $B$ to $A$).