8.311: Electromagnetic Theory  Problem Set # 2  Due: 2/19/04

Conservation laws. Magnetic dipole.

Reading: Schwinger, Chap. 3, 4, and Chap. 28 (or Jackson, Chap. 5, 6)

1. Angular momentum conservation, Schwinger, Prob. 3.5
Show that the angular momentum conservation law for the electromagnetic field can be written as
\[ \frac{\partial}{\partial t} T + \nabla \cdot K + \mathbf{r} \times \mathbf{f} = 0 \]  
(1)
where \( \mathbf{f} \) is the Lorentz force. Here the angular momentum density is \( T = \mathbf{r} \times \mathbf{G} \) and the angular momentum flux tensor is defined in terms of Maxwell stress tensor \( T_{ij} \) as
\[ K = -T \times \mathbf{r} \]
where the cross product refers to the second index of \( T_{ij} \).

2. Schwinger, Probs. 2.1, 2.2
a) Write Maxwell’s equations with magnetic charge
\[ \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E} + \frac{4\pi}{c} \mathbf{j}_e, \quad \nabla \cdot \mathbf{B} = 4\pi \rho_m \]  
(2)
\[ -\nabla \times \mathbf{E} = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{B} + \frac{4\pi}{c} \mathbf{j}_m, \quad \nabla \cdot \mathbf{E} = 4\pi \rho_e \]  
(3)
in terms of a complex vector filed \( \mathbf{F} = \mathbf{E} + i\mathbf{B} \), and related combinations of charge and current. Verify that the equations retain their form under the transformation
\[ \mathbf{F} \rightarrow e^{-i\phi} \mathbf{F} \]
where \( \phi \) is an arbitrary constant. Express this as a transformation of \( \mathbf{E}, \mathbf{B} \) and the charge-current quantities. What is the geometric interpretation? What is the particular form of this transformation when \( \phi = \pi/2 \)?

b) Suppose every charged particle carried electric and magnetic charge in the universal ratio \( g_k/e_k = \lambda \). Is there another way of looking at this situation in which we would be unaware of magnetic charges?

3. Schwinger, Prob. 3.7
As in Problem 2, let \( \mathbf{F} = \mathbf{E} + i\mathbf{B}, \mathbf{F}^* = \mathbf{E} - i\mathbf{B} \). Identify the scalar \( \frac{1}{8\pi} \mathbf{F} \cdot \mathbf{F}^* \) the vector \( \frac{1}{8\pi} \mathbf{F} \times \mathbf{F}^* \) and the tensor
\[ \frac{1}{8\pi} (\mathbf{FF}^* + \mathbf{F}^* \mathbf{F})_{ij} = \frac{1}{8\pi} \left( F_i F_j^* + F_i^* F_j \right) \]
What happens to these quantities if \( \mathbf{F} \) is replaced by \( e^{-i\phi} \mathbf{F} \)?

4. Magnetic dipole (force and torque)
a) Consider a wire loop of an arbitrary shape, carrying current \( I \) and placed in a uniform external magnetic field \( \mathbf{B} \). Find the total force and torque on the loop. Express the answer through the magnetic dipole \( \mathbf{m} = \frac{1}{2\pi} I \oint \mathbf{r} \times d\mathbf{l} \) of the loop.
b) Consider the loop of part a) in a weakly nonuniform field

\[ \mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \nabla \mathbf{B}|_{\mathbf{r}=\mathbf{r}_0} + O((\mathbf{r} - \mathbf{r}_0)^2) \]

where \( \mathbf{r}_0 \) is chosen near the loop center. Find the total force on the loop, and express it through the loop dipole moment \( \mathbf{m} \).

5. Magnetic dipole (field and interaction)
a) Show that the magnetic field of the current loop of Problem 4 at distances much larger than the loop size is given by the magnetic dipole formula

\[ \mathbf{B}(\mathbf{r}) = \frac{3\mathbf{r}(\mathbf{r} \cdot \mathbf{m}) - \mathbf{m}}{|\mathbf{r}|^3} \]  

(4)

b) Using the result of part a) or otherwise, find the potential energy of interaction between two magnetic dipoles \( \mathbf{m}_1, \mathbf{m}_2 \) located at \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \), respectively.

6. Rotating sphere
A spherical shell of radius \( a \) carries charge \( q \) which is distributed uniformly over the surface. The sphere is rotating about the \( z \) axis with an angular velocity \( \omega \).
a) Find the current density \( \mathbf{j} \) and the magnetic moment \( \mathbf{m} \) of the sphere.
b) Write down the equations for the magnetic field \( \mathbf{B} \) inside and outside the sphere and the conditions at the boundary \( |\mathbf{r}| = a \) relating the field in the inner and outer regions. Find the magnetic field in the entire space. (Hint: Assume that the field is uniform inside and of a dipole form (4) outside and match the inner and outer field values at the boundary)
c) Relate the field outside the sphere to the sphere magnetic dipole moment \( \mathbf{m} \).
d) Find the electromagnetic angular momentum of the system.

7. (Optional problem) Electric and magnetic charge system, Schwinger, Prob. 3.8
a) Electric charge \( e \) is located at the fixed point \( \frac{1}{2} \mathbf{R} \). Magnetic charge \( g \) is stationed at the fixed point \( -\frac{1}{2} \mathbf{R} \). Write down the momentum density \( \mathbf{G} \) at an arbitrary point \( \mathbf{r} \). Verify that it is divergenceless by writing it a a curl.
b) Evaluate the electromagnetic angular momentum \( \mathcal{T}_{total} = \int \mathbf{r} \times \mathbf{G} \, d^3r \). Recognize that it is a gradient with respect to \( \mathbf{R} \). Continue the evaluation to discover that it depends only on the direction of \( \mathbf{R} \), not its magnitude. This is the naive, semiclassical basis for the charge quantization condition of Dirac, \( eg = \frac{n}{2} hc \).