Physics 8.321, Fall 2002

Homework #4

Due Wednesday, October 9 by 4:30 PM in the 8.321 homework box in 4-339B.

1. Sakurai: Problem 21, Chapter 1 (page 64)

2. Sakurai: Problem 22, Chapter 1 (page 64)

3. Let $H = \frac{p^2}{2m} + V(x)$ be the Hamiltonian for a one-dimensional quantum system with discrete eigenstates $H|\alpha\rangle = E_\alpha|\alpha\rangle$. Show the following results:
   
   (a) $\sum_{\alpha} \langle \alpha | x | \alpha' \rangle^2 (E_{\alpha'} - E_\alpha) = \frac{\hbar^2}{2m}$. 
   (b) $\langle \alpha | p | \alpha' \rangle = \frac{i m \hbar}{\epsilon} \langle E_\alpha - E_{\alpha'} \rangle \langle \alpha | x | \alpha' \rangle$
   and hence $\sum_{\alpha} \langle \alpha | x | \alpha' \rangle^2 (E_{\alpha'} - E_\alpha)^2 = \frac{\hbar^2}{m^2} \langle \alpha | p^2 | \alpha \rangle$. 
   (c) Generalize to 3 dimensions and show the quantum virial theorem
   $\langle \alpha | \frac{p^2}{2m} | \alpha \rangle = \frac{1}{2} \langle \alpha | x \cdot \nabla V(x) \rangle | \alpha \rangle$.

4. A particle of mass $m$ is in a 1D potential $V(x) = v\delta(x-a) + v\delta(x+a)$ where $v < 0$.
   (a) Find the wave function for a bound state with even parity ($\psi(x) = \psi(-x)$).
   (b) Find an expression for the energy for even parity states, and determine how many such states exist.
   (c) Solve for the even parity bound state energy when $\frac{ma^2v}{\hbar^2} \ll 1$.
   (d) Repeat parts (a) and (b) for odd parity ($\psi(x) = -\psi(-x)$). For what values of $v$ are there bound states?
   (e) Find the even and odd parity state binding energies for $\frac{ma^2v}{\hbar^2} \gg 1$, and explain physically why these energies move closer together as $a \to \infty$.

5. Define the coherent state $|\phi\rangle = e^{\phi a^\dagger} |0\rangle$, where $\phi$ is a complex number, $a^\dagger$ is the creation operator for a harmonic oscillator, and $|0\rangle$ is the oscillator ground state. Show that $|\phi\rangle$ has the following properties:
   (a) $|\phi\rangle = \sum \frac{\phi^n}{\sqrt{n!}} |n\rangle$
   (b) $a|\phi\rangle = \phi|\phi\rangle$
   (c) $\langle \phi | \phi' \rangle = e^{\phi^* \phi'}$
   (d) $\langle \phi : A(a^\dagger, a) : \phi' \rangle = e^{\phi^* \phi'} A(\phi^*, \phi')$, where $: A(a^\dagger, a) :$ is “normal ordered” so that all creation operators $a^\dagger$ are to the left of all annihilation operators $a$.
   (e) $\int \frac{d\phi^* d\phi}{2\pi i} e^{-\phi^* \phi} |\phi\rangle \langle \phi | = 1$. (completeness for coherent states)
6. Define a *squeezed state* to be a state of the form

\[
|\alpha, \beta, \gamma\rangle = e^{\alpha^2 + \beta a^+ + \gamma (a^+)^2} |0\rangle
\]

in the single harmonic oscillator Hilbert space

(a) Compute the norm \(\langle \alpha, \beta, \gamma | \alpha, \beta, \gamma \rangle\) in the special case \(\beta = 0\). What is the condition needed for this norm to be finite? Can you generalize your result to \(\beta \neq 0\)?

(b) Show that the position basis state \(|x'\rangle\) can be written in the form (1), and find the associated values \(\alpha(x'), \beta(x'), \gamma(x')\). Does your expression for \(|x'\rangle\) give a state of finite norm in the Hilbert space?

(c) Use your answer to (b) to give squeezed state descriptions of the kets associated with the wavefunctions \(\psi(x') = \delta(x')\) and \(\psi(x') = 1\).

(d) Describe the kets associated with the wavefunctions \(\delta(x' \pm y')\) in squeezed state form

\[
\exp \left[ F(a^+_x, a^+_y) \right] (|0\rangle_x \otimes |0\rangle_y)
\]

where \(F\) is a quadratic function of \(a^+_x, a^+_y\).