1. Minimum uncertainty states (adapted from Sakurai 1.18)

(a) The simplest way to derive the Schwarz inequality goes as follows. First observe

\[(\langle \alpha \mid + \lambda^* \langle \beta \mid \cdot (\mid \alpha \rangle + \lambda \mid \beta \rangle) \geq 0 \]  \hspace{1cm} (1)

for any complex number \(\lambda\); then choose \(\lambda\) in such a way that the preceding inequality reduces to the Schwarz inequality.

(b) Show that the equality sign in the generalized uncertainty relation holds if the state in question satisfies

\[\Delta A \mid \alpha \rangle = \lambda \Delta B \mid \alpha \rangle \]  \hspace{1cm} (2)

with \(\lambda\) purely imaginary.

(c) Hence show that the Gaussian wave packet

\[\langle x'\mid \alpha \rangle = (2\pi d^2)^{-1/4} \exp \left( \frac{i\langle p \rangle x' - (x' - \langle x \rangle)^2}{\hbar} \right) \]  \hspace{1cm} (3)

satisfies the minimum uncertainty relation for \(x\) and \(p\).

2. The uncertainty relation for spin (Sakurai 1.19)

3. Sakurai 1.22
4. (a) For a particle in a state described by a real wave function show that the average momentum \( \langle p \rangle = 0 \).

(b) Suppose the particle is in a general state with wave function \( \psi(x) \) with average momentum \( p_0 \). Show that the modified wave function \( e^{i\frac{Px}{\hbar}}\psi(x) \) has average momentum \( p_0 + P \).

(c) A different perspective on the result above is obtained by considering the operator \( \tilde{T}(P) = e^{\frac{iPx}{\hbar}} \) where \( x \) is the position operator and \( P \) is real. Find \( \tilde{T}(P)p\tilde{T}^\dagger(P) \) and \( \tilde{T}(P)|p'\rangle \). Argue how the result of (b) follows from this calculation.

(d) Consider \( \tilde{T}(P) \) together with the translation operator \( T(a) = e^{-\frac{ia}{\hbar}} \) defined in class. Is the product \( \tilde{T}(P)T(a) \) equal to \( e^{\frac{i(Px-pa)}{\hbar}} \)?

5. Consider a spin-1/2 particle in the presence of a time independent magnetic field along the z-direction with the Hamiltonian

\[
H = -\gamma BS_z
\]  

(\( \gamma \) is a constant). At time \( t = 0 \) the spin state is an eigenstate of \( \vec{S} \cdot \hat{n}_0 \) where \( \hat{n}_0 \) lies in the \( xz \) plane and makes an angle \( \theta \) with the \( z \) axis. Here you will consider the time evolution of this state.

(a) In the Schrodinger picture find the state at time \( t \) in the \( S_z \) basis. Hence find the probability that a measurement of \( S_x \) at time \( t \) will yield the value \( +\frac{\hbar}{2} \).

(b) Recall that any state of a spin-1/2 system can be represented as a point \( \hat{n} \) in the Bloch sphere. The evolution of the spin state in this problem can be described as a trajectory \( \hat{n}(t) \) on the Bloch sphere with \( \hat{n}(t = 0) = \hat{n}_0 \). Show explicitly using the result above that \( \hat{n}(t) \) precesses about the \( z \)-axis at the Larmor frequency \( \gamma B \).

(c) Now using the Heisenberg picture derive the equation of motion for \( \langle \vec{S} \rangle \). Re-derive the same equation within the Schrodinger picture using the calculation above.