1. Three spins A, B, and C - each with magnitude $1/2$ - are prepared in a state

$$|\psi\rangle = N (|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\uparrow\uparrow\rangle)$$  \hspace{1cm} (1)

The first spin describes A, the second B, and the third C, all in the $S_z$ basis.

(a) Determine $N$ so that the state is normalized.
(b) Find the probability that a measurement of $S_z$ for A gives $+\frac{\hbar}{2}$.
   Repeat for the same measurement for either B or for C.
(c) Calculate the density matrix of A.
(d) Use (c) to obtain the probabilities of the outcome of $S_z$ measurements on A. Repeat for $S_x$ measurements on A.

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3. Consider two quantum spins in a state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0, +\rangle + |1, 0\rangle)$$  \hspace{1cm} (2)

Here 0, 1 refer to spin eigenstates in the $z$ basis and ± to spin eigenstates in the $x$ basis. Find the Schmidt decomposition for this state, and the density matrix for each of the two spins.

4. Consider a quantum system $A$ in a mixed state described by a density matrix $\rho_A$. It is possible to regard this mixed state as arising due to entanglement of $A$ with some other system $B$ such that the composite system $A + B$ is in a pure state $|\psi_{A+B}\rangle$. The state $|\psi_{A+B}\rangle$ is known as a ‘purification’ of $\rho_A$ in B and satisfies
The purification is clearly not unique. Show that any two purifications of $\rho_A$ on $B$ are related by a unitary operator acting only on subsystem $B$. In other words two different purifications $|\psi_{A+B}\rangle$ and $|\psi'_{A+B}\rangle$ satisfy

$$|\psi'_{A+B}\rangle = U_B |\psi_{A+B}\rangle$$

(4)

5. von Neumann entropy

The von Neumann entropy of a system described by a density matrix $\rho$ is defined as

$$S = -Tr \rho \ln \rho$$

(5)

Prove the following properties of the von Neumann entropy of a quantum system.

(a) The von Neumann entropy is invariant under unitary transformations of the density matrix $\rho$.

(b) If $\rho$ has $D$ non-vanishing eigenvalues then

$$S(\rho) \leq \log D$$

(6)

with equality only when all the non-zero eigenvalues are equal. This means that the entropy is maximized when the quantum state is chosen randomly.

(c) If $\rho_1$ and $\rho_2$ describe two density matrices for a system, then first show that $\rho(\lambda) = \lambda \rho_1 + (1 - \lambda) \rho_2$ is also a legitimate density matrix. Next show that

$$S(\rho(\lambda)) \geq \lambda S(\rho_1) + (1 - \lambda) S(\rho_2)$$

(7)

This means that the more ignorant we are about how the state was prepared the higher the von Neumann entropy.

(d) For a composite system $A + B$ in a general mixed state described by density matrix $\rho_{A+B}$,

$$S(\rho_{A+B}) \leq S(\rho_A) + S(\rho_B)$$

(8)

with equality for uncorrelated systems where the full density matrix factorizes.