Quantum Field Theory II (8.324) Fall 2010
Assignment 1

Readings

- Peskin & Schroeder chapters 15 and 16
- Weinberg vol 2 chapter 15.
- Prof. Zwiebach's notes on Lie algebras

Note:

- In lectures I will focus on presenting physical ideas and will not have time to introduce mathematical background extensively. For relevant background on Lie algebras and their representations please read Prof. Zwiebach's notes I posted on the web and Peskin & Schroeder's section 15.4 “Basic facts about Lie Algebras”.
- There are deep connections between the geometric structure of Yang-Mills theory and Einstein’s general relativity. For those of you who have studied general relativity should read the end of sec. 15.1 and sec. 15.3 of Weinberg Volume II.
- There is a rich mathematical structure behind Yang-Mills theory carrying the name fibre bundle. Those of you who are interested in digging into this a little deeper can find a nice description in the book by John Baez and Javier Muniain, “Gauge fields, knots and gravity”, World Scientific (1994).

This paper also review attempts to unify gauge theories and general relativity. At almost the same time of the paper of Yang-Mills the theory was also discovered independently by Ron Shaw who wrote it in his Cambridge University Ph.D thesis, but never published it. Shaw later became a Mathematician at Hull University, UK. Some of you might find it interesting to read about Shaw’s discovery in the reminiscence of Shaw himself:
http://www.hull.ac.uk/php/masrs/reminiscences.html
Problem Set 1

1. Non-Abelian global symmetries and the associated charges (30 points)

Consider the following Lagrangian

\[ \mathcal{L} = -i \bar{\Psi} \gamma^{\mu} \partial_{\mu} \Psi - m \Psi \]  

(1)

where

\[ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \bar{\Psi} = \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix} \]  

(2)

\( \psi_{1,2} \) are Dirac spinor fields. We will suppress spinor indices throughout.

(a) Show that (1) is invariant under infinitesimal transformations

\[ \delta \Psi = i \epsilon_a T_a \Psi, \quad T_a = \frac{\sigma_a}{2}, \quad a = 1, 2, 3 \]  

(3)

where \( \sigma_a \) are Pauli matrices.

(b) Find the conserved currents \( J_a^\mu \) corresponding to the symmetric transformations (3).

(c) Write down the corresponding conserved charges \( Q_a, a = 1, 2, 3 \). Show that

\[ \delta \Psi = i [\epsilon_a Q_a, \Psi] \]  

(4)

(d) Find the commutation relations between \( Q_a \)'s (using the canonical commutators).

(e) Let

\[ \hat{U} = \exp (i \Lambda_a Q_a), \quad U = \exp (i \Lambda_a T_a) \]  

(5)

for some constants \( \Lambda_a \). Note that \( \hat{U} \) is a quantum operator, while \( U \) is a 2 \( \times \) 2 unitary matrix. Show that

\[ \hat{U} \Psi \hat{U}^\dagger = U \Psi \]  

(6)

2. Parallel transport around a small loop (20 points)

Consider a small closed loop \( C \) which we take to be a parallelogram with one corner at \( x^\mu \) and two sides \( a^\mu \) and \( b^\mu \). Denote the transport around the loop to be \( U_C(x, x) \). Show that \( U_C(x, x) \) can be expressed in terms of the field strength in both \( U(1) \) and general non-Abelian case. You should write down the explicit expressions of \( U_C(x, x) \) in terms of the field strength. State how this result can be generalized to an arbitrary small closed loop.
3. Bianchi identity (15 points)

Check the Bianchi identity

\[ D_\mu F_{\nu\lambda} + D_\lambda F_{\mu\nu} + D_\nu F_{\lambda\mu} = 0 \]  \hspace{1cm} (7)

where

\[ D_\mu F_{\nu\lambda} \equiv \partial_\mu F_{\nu\lambda} - ig[A_\mu, F_{\nu\lambda}] \]  \hspace{1cm} (8)

4. Scalar propagator in a gauge theory (35 points)

(a) Peskin & Schroeder prob. 15.4 part (a)

(b) Peskin & Schroeder prob. 15.4 part (b)

(c) Consider \( n \) complex scalar fields of mass \( m \) arranged in a vector

\[ \Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} \]  \hspace{1cm} (9)

Write down a \( U(n) \) gauge invariant Lagrangian for \( \Phi \). You can assume that \( \Phi \) has no self-interactions.

(d) Peskin & Schroeder prob. 15.4 (c) (using the Lagrangian you obtained in part (c) above).
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