5.3.2: Computation of Beta-Functions

We consider the beta functions for the mass \( m \) and the coupling \( g \):

\[
\beta_g = \mu \frac{dg}{d\mu}, \quad \beta_m = \mu \frac{d\lambda_m}{d\mu},
\]

where \( \lambda_m = \frac{m^2(\mu)}{\mu^2} \). Note that the coupling constants in different renormalization schemes are generally different. In general, we have

\[
\{ \lambda_i \} : \text{ scheme 1,} \\
\{ \tilde{\lambda}_i \} : \text{ scheme 2.}
\]

In the problem set, we will see how the \( \beta \)–functions transform. In particular, the first two terms are universal.

**Example 1:** \( g\phi^3 \) in \( d = 6 \) with MS scheme

\[
\mathcal{L} = -\frac{1}{2}(\partial \phi_B)^2 - \frac{1}{2}m_B^2 \phi_B^2 + \frac{g_B}{6} \phi_B^3
\]

\[
= -\frac{1}{2}(1 + A) (\partial \phi)^2 - \frac{1}{2}m^2(1 + B) \phi^2 + \frac{g}{6} \mu^2 (1 + C) \phi^3,
\]

with

\[
\phi_B = (1 + A)^\frac{3}{2} \phi, \\
m_B = (1 + A)^{-\frac{1}{2}} (1 + B)^\frac{3}{2} m, \\
g_B = g\mu^\frac{3}{2} (1 + A)^\frac{3}{2} (1 + C),
\]

\( A = -\frac{2}{d-6} \), \( B = -\frac{2}{d-6} \) and \( C = -\frac{2}{d-6} \), up to \( O(\alpha^2) \). The key is to note that the bare quantities should be independent of \( \mu \):

\[
\mu \frac{dm_B}{d\mu} = 0, \quad \mu \frac{dg_B}{d\mu} = 0.
\]

This leads to results for \( \beta_g \) and \( \beta_m \). In general, in the case of dimensional regularization and minimal subtraction,

\[
g_i^{(B)} = \mu^{\delta_i(\epsilon)} \left[ \lambda_i(\mu) + \sum_{n=1}^{\infty} \epsilon^{-n} G_i^{(n)}(\lambda_j) \right]
\]

where \( \delta_i(\epsilon) = \delta_i + a_i \epsilon \): the last correction is due to dimensional regularization. From (2), we have

\[
\beta_i(\epsilon) = \mu \frac{d\lambda_i}{d\mu}
\]

We can expand

\[
\beta_i(\epsilon) = \beta_i + \epsilon \alpha_i
\]

where the first term is the \( \beta \)–function and the second term again comes from dimensional regularization. If we take the \( \mu \)–derivative of (3), we find

\[
0 = \delta_i(\epsilon) \left[ \lambda_i + \sum_{n=1}^{\infty} \epsilon^{-n} G_i^{(n)} \right] + \left[ \beta_i(\epsilon) + \sum_{n=1}^{\infty} \epsilon^{-n} \frac{\partial G_i^{(n)}}{\partial \lambda_j} \beta_j(\epsilon) \right].
\]
Equating both sides of the above equation order by order in $\epsilon$, we find
\[
(\delta_i + a_i \epsilon) \left[ \lambda_i + \epsilon^{-1} G_i^{(0)} + \epsilon^{-2} G_i^{(2)} + \ldots \right] + \left[ \beta_i + \epsilon \alpha_i + \epsilon^{-1} \frac{\partial G_i^{(0)}}{\partial \lambda_j} (\beta_j + \epsilon \alpha_j) + \epsilon^{-2} \ldots \right] = 0,
\]
and so, at $O(\epsilon)$,
\[
\alpha_i = -\lambda_i a_i,
\]
(note that we are not invoking the summation convention here.) and, at $O(\epsilon^2)$,
\[
\delta_i \lambda_i + a_i G_i^{(1)} + \beta_i + \sum_j \alpha_j \frac{\partial G_i^{(1)}}{\partial \lambda_j} = 0,
\]
or, equivalently,
\[
\beta_i = -\delta_i \lambda_i - a_i G_i^{(1)} + \sum_j \frac{\partial G_i^{(1)}}{\partial \lambda_j} a_j \lambda_j.
\]
We note that the $\beta_i$ are determined by simple-pole residues of the counter-terms, and that at $O(\epsilon^{-n})$ for $n \geq 1$, the constraints determine $G_i^{(n)}$ for $n \geq 2$ in terms of $G_i^{(1)}$. We now return to our discussion of the $g\phi^3$–theory. Here,
\[
g_1 = \alpha_B = \alpha \mu^2 (1 + A)^{-3} (1 + B), \quad g_2 = m_B^2 = \mu^2 \lambda_m \left( 1 - \frac{5 \alpha}{6 \epsilon} + \ldots \right),
\]
where $\alpha \equiv \frac{\sigma^2}{(4\pi)^2}$, and so we find
\[
\delta_1 (\epsilon) = \epsilon, \quad \delta_1 = 0, \quad a_1 = 1, \quad G_1^{(1)} = -\frac{3}{2} \alpha^2,
\]
and
\[
\delta_2 (\epsilon) = 2, \quad \delta_2 = 2, \quad a_2 = 0, \quad G_1^{(1)} = -\frac{5}{6} \lambda_m \alpha.
\]
From this, we find for the $\beta$–functions,
\[
\beta_\alpha = \frac{3}{2} \alpha^2 - 3 \alpha^2 = -\frac{3}{2} \alpha^2, \\
\beta_m = -2 \lambda_m - \frac{5}{6} \lambda_m \alpha = -2 \lambda_m - \frac{5}{6} \lambda_m \alpha + \ldots.
\]
Let us consider the physical implications of these equations.

1. At weak coupling, $\alpha \ll 1$, $\beta_m$ is dominated by the first term, $\beta_m \approx -2 \lambda_m$. This gives the dimension in the absence of the interaction, which implies the familiar behaviour
\[
\lambda_m (\mu) = \lambda_m (\mu_0) \left( \frac{\mu_0}{\mu} \right)^2,
\]
and so $\lambda_m (\mu)$ grows quadratically as we decrease $\mu$.

2. $\alpha$ is marginal in the absence of interactions, and so, interactions are important to determine the leading contribution. For $g\phi^3$, $\beta_2 < 0$, and the coupling is marginally relevant: $\alpha$ becomes stronger going into the infrared, as we decrease $\mu$, and stronger going into the ultraviolet, as we increase $\mu$.

We now integrate
\[
\frac{\mu \, d\alpha}{d\mu} = -\frac{3}{2} \alpha^2,
\]
which is equivalent to
\[
d \left( \frac{1}{\alpha} \right) = \frac{3}{2} d \log \mu.
\]
Suppose that $\alpha(\mu_0) = \alpha_0$. Then we have

$$\frac{1}{\alpha(\mu)} - \frac{1}{\alpha_0} = \frac{3}{2} \log \frac{\mu}{\mu_0}, \quad (14)$$

and hence,

$$\alpha(\mu) = \frac{\alpha_0}{1 + \frac{3\alpha_0}{2} \log \frac{\mu}{\mu_0}}. \quad (15)$$

In particular, as $\mu \to \infty$, $\alpha(\mu) \to 0$. This is asymptotic freedom. $\alpha(\mu) \to \infty$ when $\frac{3\alpha_0}{2} \log \frac{\mu}{\mu_0} = -1$. That

$$\frac{1}{\alpha(\mu)}$$

log($\mu$)

![Figure 1](image1.png)

Figure 1: We find a linear relationship between $\alpha^{-1}(\mu)$ and $\log(\mu)$ with a positive gradient for the $\phi^3$ theory.

is, when $\mu = \mu_0 e^{-\frac{m^2}{\mu_0^2}} \equiv \Lambda$. We note that this discussion only applies to $\alpha(\mu) \ll 1$. Of course, our one-loop approximation already breaks down before $\Lambda$ is reached. Nevertheless, $\Lambda$ provides a characteristic scale for the system. $\Lambda$ is independent of $\mu_0$. We can rewrite

$$\alpha(\mu) = \frac{2}{3} \frac{1}{\log \frac{\mu}{\Lambda}}, \quad (16)$$

and instead of specifying $\alpha(\mu_0) = \alpha_0$, we can simply specify $\Lambda$. The system does not have any dimensionless coupling. Rather, it has only a scale $\Lambda$. This is known as **dimensional transmutation**.

Now let us go back to the issue of the large logarithms encountered in the on-shell scheme when $k^2 \gg m^2$. We encountered $\alpha \log \frac{k^2}{m^2}$ at the one-loop level. However, it goes away if we choose $\mu \sim k$. We want to understand why this is, and why we can still trust perturbation theory. To see what is happening, let us consider, for $\mu \sim k$

$$\alpha(\mu_0 = m) = \alpha_0, \quad (17)$$

so that

$$\alpha(\mu) = \frac{\alpha}{1 + \frac{3}{2} \alpha_0 \log \frac{\mu}{m}} \sim \alpha_0 \sum_{n=0}^{\infty} \left( \alpha_0 \log \frac{\mu}{m} \right)^n. \quad (18)$$

These are exactly the logarithmic terms we have seen before. They were just transferred to the relation between $\alpha(\mu \sim k)$ and $\alpha_0$. The higher loop correction give higher powers in $\alpha_0 \log \frac{\mu}{m}$. As a perturbation series in $\alpha_0$, the last expression becomes bad when $\alpha_0 \log \frac{\mu}{m}$ becomes large, but through the miracle of the renormalization group flow, by integrating the $\beta-$function, we have essentially resummed this bad series as far as $\alpha(\mu)$ remains small for all $\mu$. This remarkable result is the essence of the renormalization group flow, which clearly also applies to the Wilsonian approach.

![Figure 2](image2.png)

Figure 2: $\alpha(\mu)$ and $\alpha(\mu_0)$ are separated by large logarithms, but if we take infinitesimal steps, $\Delta \alpha = \alpha^2(\mu) \frac{\Delta \mu}{\mu} \sim \alpha^2(\mu) \frac{\Delta \mu}{\mu}$, and so for $\alpha^2(\mu)$ small, we can ignore the higher order terms.