# Chapter 9

# Resolution of $U_A(1)$

Resolution of the $U_A(1)$ problem: We have apparently,

$$\partial_\mu j^{\mu 5} \approx \epsilon^{\alpha\beta\gamma\delta} G^a_{\alpha\beta} G^a_{\gamma\delta} (+mass\ terms) \tag{9.1}$$

so that symmetry is removed. However, for the Abelian case we saw that the right-hand side is a total derivative, and this continues to be true in the non-Abelian case. Indeed

$$\begin{aligned} \epsilon^{\alpha\beta\gamma\delta}(\partial_\alpha A^a_\beta + f^{abc} A^b_\alpha A^c_\beta)(\partial_\gamma A^a_\delta + f^{ade} A^d_\gamma A^e_\delta) &= \partial_\alpha(\epsilon^{\alpha\beta\gamma\delta}(A^a_\beta \partial_\gamma A^a_\delta + \frac{2}{3} f^{abc} A^a_\beta A^b_\gamma A^c_\delta)) \\ &\equiv \partial_\alpha K^\alpha \end{aligned} \tag{9.2}$$

So there is a modified current that generates a legitimate symmetry.

This is not necessary a disacter, because the current is not gauge-invariant. So the massless states it produces might not be physical (that's what happens in the Meissner-Higgs effect), but this is not convincing unless we have a mechanism.

Here is the basic story-line:

1. Since $K^\alpha$ is not gauge-invariant, it does not appear directly in the action, and it may fluctuate more wildly. This could prevent the surface terms in

$$\int \partial j - \int \epsilon GG = \int \partial K = \int_S K dS \tag{9.3}$$

   from vanishing.

2. Concretely, we can imagine $A \sim\uparrow^\infty_r \frac{1}{r}$ in such a way that $K \sim \frac{1}{r^3}$ but $G^2$ falls off faster.

3. In fact this will happen if $A$ approaches a non-trivial (direction dependent) pure gauge configuration.

$$A_\alpha \to \Omega^{-1}\partial_\alpha \Omega \tag{9.4}$$

4. Specifically, the integral $\int K^\alpha dS^\alpha$ is proportional to the degree of the mapping from $S^3$, the boundary of (Euclideanized) space-time into the manifold of $SU(2)$, which is also topologically $S^3$.

   Mathematical aside: $SU(2)$ matrices: $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $|\alpha|^2 + |\beta|^2 = 1 \Rightarrow S^3$. Other groups: deform map into a sphere $\pi^3(G) = Z$ for any group $G$.

5. From the inequality

$$\int tr(G_{\mu\nu} - \tilde{G}_{\mu\nu})^2 \geq 0 \tag{9.5}$$

   (with $\tilde{G}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}$) we derive

$$\int trG_{\mu\nu}G_{\mu\nu} \geq |\int trG_{\mu\nu}\tilde{G}_{\mu\nu}| \quad (\geq 8\pi^2) \tag{9.6}$$

   Thus the weight $e^{-\frac{1}{g^2}\int GG}$ of configurations that violate the conservation law is $\leq e^{-\frac{8\pi^2}{g^2}}$. It vanishes to all nodes in perturbation theory.

   Scholium:

   As in the Higgs effect the massless would be Nambu-Goldstone modes are subject to long-range forces involving gauge fields, but here the relevant gauge fluctuations are topological and quantized. Because of this, the mass-generation mechanism is essentially nonperturbative. Of course, this had to be so in QCD, since the only mass scale is $\sim e^{-\frac{8\pi^2}{g^2}}$.

6. The bound is saturated when

$$G_{\mu\nu} = \pm\tilde{G}_{\mu\nu} \tag{9.7}$$

   This can be solved fairly simply for the minimal charges, and using very elaborate mathematics for higher charges. However, the precise solutions are limited value for practical QCD, basically, because the coupling is just uniformly small.

7. The detailed way in which $U_A(1)$ violation shows up – in terms of changes in fermion quantum numbers or occurance of otherwise forbidden processes, is quite interesting and important in other contexts.

   When calculating a transition amplitude using path integrals, one must calculate the fermion determinant. The integrated anomaly equation

$$\Delta Q_5 = (\Delta winding) \times (\# \; doublets) \tag{9.8}$$

   suggests that this determinant will vanish when $\Delta winding$ is evaluated between states in which the $\Delta Q_5$ is not appropriate. This signals zero-energy solution, a zero-modes of the Dirac equation. There is a corresponding mathematical theorem, the Atiyah-Singer index theorem.

8. To evaluate processes in the nonvanishing sector, we must allow for creation and destruction of fermions. This is done by coupling in source and taking derivatives, e.g.,

$$\begin{aligned} < \bar{\psi}_{x_1} \bar{\psi}_{x_2} \psi_{x_3} \psi_{x_4} > \;\; \sim \;\; & \frac{\delta}{\delta\eta(x_1)} \frac{\delta}{\delta\eta(x_2)} \frac{\delta}{\delta\bar{\eta}(x_3)} \frac{\delta}{\delta\bar{\eta}(x_4)} \Big|_{\eta=0} \\ & \times \frac{\int e^{-(S_{gauge} + \bar{\psi}\nabla\psi + \bar{\eta}\psi + \bar{\psi}\eta)}}{\int e^{-(S_{gauge} + \bar{\psi}\nabla\psi)}} \end{aligned} \tag{9.9}$$

   The zero-modes give $\nabla\psi_0 \sim \eta\psi_0$ and so directly a prove of $\eta$ (or $\bar{\eta}$) in the determinant. We get a nonzero answer by solving them up with $\frac{\delta}{\delta\eta}$ factor. This enforces the anomaly equation (for more algebraic details see Coleman).

   Scholium:

   In the context of electroweak theory, the $SU(2)_L$ anomaly equation suggests a mechanism of baryon number violation. I have asked you to spell this out as a problem.