Problem 1) Composite Operator Renormalization

In lecture we discussed the renormalization of parameters in a Lagrangian. Often one is interested in studying a local current built from a product of fields at the same space-time point. For example, consider $J_{\mu\nu} = \bar{\psi}(0)\sigma_{\mu\nu}\psi(0)$ in QED. External operators like $J_{\mu\nu}$ may require renormalization beyond that associated with the field $\psi(0)$ (you discussed an example of this type in section). In this problem I want you to compute the anomalous dimension for $J_{\mu\nu}$. To turn this into a problem you know how to solve, consider adding a term $\mathcal{L}_{\text{int}} = C^\mu\nu_0 \bar{\psi}_0 \sigma_{\mu\nu} \psi_0$ (1)

to the QED Lagrangian. By switching from bare to renormalized coefficients and fields as usual, compute the anomalous dimension for $C^\mu\nu_0$ in MS to order $\epsilon^2$. [Hint: Remember you only need the $1/\epsilon$ poles. Compare eqs.18.10 and 18.11 in Peskin. You can use $Z_\psi$ from Peskin without computation.] Now write $\mathcal{L}_{\text{int}} = C^\mu\nu(\mu)J_{\mu\nu}(\mu)$ to define the renormalized current, and use the anomalous dimension for $C^\mu\nu(\mu)$ to find the anomalous dimension of $J_{\mu\nu}(\mu)$.

Problem 2) The decay $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$

At low energy, $p^2 \ll m_W^2 = (80 \text{ GeV})^2$ we can expand the $W$-boson propagator, $1/(p^2 - m_W^2) = -1/m_W^2 + \ldots$. Keeping the first term gives the effective four-fermion interaction

$$\mathcal{L}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{ud} [\bar{d}\gamma^\alpha(1 - \gamma_5)u][\bar{\nu}_\mu\gamma_\alpha(1 - \gamma_5)\mu] + \text{h.c.}$$ (2)

which allows $\bar{u}d \rightarrow \mu^- \bar{\nu}_\mu$ and hence $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$. Define the decay constant for the pion as

$$\langle 0 | \bar{u}\gamma^\alpha\gamma_5 d | \pi^-(p_\pi) \rangle = -i f_\pi p_\pi^\alpha,$$ (3)

and show that the rate for $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$ is

$$\Gamma = \frac{G_F^2}{8\pi} |V_{ud}|^2 m_\mu^2 m_\pi f_\pi^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2.$$ (4)

Comparing with data, and taking $V_{ud} \simeq 1$, determine a value for $f_\pi$. There are two common normalization conventions for $f_\pi$ in the literature. The one in this problem agrees with the convention in the chiral Lagrangian in problem 3 below, $f_\pi = f$. The other convention is $\langle 0 | J_5^{\alpha,a} | \pi^+(p_\pi) \rangle = -i F_\pi F_\pi^a \delta^{ab}$ where $J_5^{\alpha,a} = \bar{\psi}\gamma^\alpha\gamma_5(\tau^a/2)\psi$. Derive the relation between $F_\pi$ and $f_\pi$. 

Problem 3) The decays $\pi^- \rightarrow \pi^0 e^- \bar{\nu}_e$ and $K^- \rightarrow \pi^0 \pi^0 e^- \bar{\nu}_e$

In lecture we derived a Feynman rule for the $\pi^-$ to $W^-$ transition by starting from the gauged SU(2) chiral Lagrangian,

$$\mathcal{L}_\chi = \frac{f^2}{8} \text{tr} [D^{\mu} \Sigma D^{\mu} \Sigma^\dagger]$$

where $D^{\mu} \Sigma = \partial^{\mu} \Sigma + i \ell^{\mu} \Sigma$ and $\ell^{\mu} = - \frac{g_2}{\sqrt{2}} W^\mu_+ T^+ + \text{h.c.}$ [for simplicity we’ll assume $V_{ud} = 1$ in this problem].

a) By carrying out this expansion to higher order in the pion fields derive the amplitude for $\pi^- \rightarrow \pi^0 e^- \bar{\nu}_e$.

b) Now generalize to the SU(3) case. Draw the tree level Feynman diagrams that can contribute to $K^- \rightarrow \pi^0 \pi^0 e^- \bar{\nu}_e$. Derive the Feynman rules that you would need to compute this amplitude (in the charged Goldstone-boson basis). Recall that $\mathcal{L}_\chi$ has a four-boson interaction. Feel free to use a program like Mathematica to take the traces. You may also use results given in lecture.