1. Classical Model of the Light Force
In class, we will discuss light forces using the OBE. In preparation for this, you are asked here to consider a (semi-)classical derivation. Assume that a hydrogenic atom can be modeled classically as an electron harmonically bound to a nucleus, with a resonant frequency $\omega_0$ and damping coefficient $\gamma$. The nucleus is fixed at position $x_0$ while the electron’s position is denoted by $x$. Now suppose the atom is illuminated with an electromagnetic wave of the form

$$E(x, t) = \hat{\epsilon}E_0(x) \cos(\theta(x) + \omega t)$$  \hspace{1cm} (1)

where $\theta(x)$ is the phase of the wave as a function of position $x$ at time $t = 0$. The dipole moment of the electron may be written as

$$p(x, t) = \hat{\epsilon}(u \cos(\theta(x) + \omega t) - v \sin(\theta(x) + \omega t))$$  \hspace{1cm} (2)

Then the force of the light on the atom is

$$F = (p \cdot \hat{\epsilon})\nabla E(x, t)$$  \hspace{1cm} (3)

(a) **Time averaged force**
Make the dipole approximation that $E(x) \approx E(x_0)$. Show that the time averaged force is

$$\langle F \rangle = \frac{1}{2}(\hat{p} \cdot \hat{\epsilon})(u \hat{\nabla} E_0(x_0) + v E_0(x_0) \nabla \theta(x_0))$$  \hspace{1cm} (4)

This expression is exactly analogous to the quantum-mechanically derived force. The first term is the dipole (stimulated) force, and the second term is the scattering (spontaneous) force.

(b) **The potential picture**
Recalculate the time averaged force on the atom from the instantaneous potential energy of a dipole in an electric field. How does this answer differ from that of 1a? Speculate as to why.

(c) **Dipole moment of electron**
Now we will solve explicitly for the dipole moment of the electron. In complex notation, the equation of motion is

$$m \frac{\partial^2 \mathbf{r}}{\partial t^2} + \gamma \frac{\partial \mathbf{r}}{\partial t} + m\omega_0^2 \mathbf{r} = -e\hat{\epsilon}E_0(x_0)e^{i(\theta(x_0) + \omega t)}$$  \hspace{1cm} (5)

where $\mathbf{r} = x - x_0$. Solve this equation to find $p = -e\mathbf{r}$. Substitute the quadrature components of $p$ into the force equation from part (a) to find that

$$F = -\frac{e^2}{2m\omega_0} \frac{\delta \nabla E_0^2 + \Gamma \nabla \theta}{4\delta^2 + \Gamma^2}$$  \hspace{1cm} (6)

where $\delta = \omega - \omega_0$ and $\Gamma = \gamma/m$. Make the approximation that $\omega \approx \omega_0$. 

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**Assignment #8**

Due: Monday, April 22
2. Master equation for a damped optical cavity

A Fabry-Perot cavity can be modeled as being made of a high reflectivity mirror and a perfect mirror with fixed spacing. Clearly, photons stored inside this cavity will gradually leak out the partially reflecting mirror, causing the state inside to change. This process is described by a master equation, much like an atom coupled to fields is described by the optical Bloch equation. In this problem, we explore a simple derivation of such a master equation, for a single mode cavity.

Let \( a \) and \( a^\dagger \) describe the optical mode of interest within the cavity, with characteristic energy \( \hbar \omega \), described by the Hamiltonian \( H_0 = \hbar \omega a^\dagger a \). Let \( |\psi\rangle \) be the initial cavity state. Let us suppose that photons leak out of the cavity at a rate proportional to the photon number in the cavity and to \( \Gamma \), which parameterizes the leakiness of the leaky mirror. Thus the probability that a photon leaks out within the infinitesimal timestep between \( t \) and \( t + dt \) is given by \( dp = \Gamma dt \langle \psi|a^\dagger a|\psi\rangle \). And with a probability \( 1 - dp \) it doesn’t. For these two cases we model the time evolution as follows:

- If a photon leaks out, the cavity state becomes \( |\tilde{\psi}_1\rangle = a\sqrt{\Gamma} d|\psi\rangle \).
- If no photon leaks out, the state becomes \( |\tilde{\psi}_0\rangle = e^{-iHt/\hbar} |\psi\rangle \) having evolved under the “Hamiltonian” \( H = H_0 - i(\hbar/2)a^\dagger a \). Note that \( H \) is not Hermitian. It has an imaginary term, so it is “lossy”. Evolving a state forward in time with \( H \) yields an un-normalized state.

The two un-normalized states \( |\tilde{\psi}_0\rangle \) and \( |\tilde{\psi}_1\rangle \) will later be two “components” of the density matrix.

a) Compute the normalized states \( |\psi_0\rangle \) and \( |\psi_1\rangle \). Provide some words of justification for the lossy term in \( H \) (for instance, why is it proportional to \( a^\dagger a \)?).

Where you encounter exponentials of \( H \) or \( H^\dagger \) (which is NOT equal to \( H \)), expand them to first order in \( dt \).

b) At time \( t \), suppose the cavity starts in the pure quantum state \( \rho(t) = |\psi\rangle \langle \psi| \). At time \( t + dt \) the state of the leaky cavity system is given by

\[ \rho(t + dt) = (1 - dp)|\psi_0\rangle \langle \psi_0| + dp|\psi_1\rangle \langle \psi_1| . \]  \hspace{1cm} (9)

Rewrite this density matrix, building it out of the the un-normalized states described in part a).

The only bras or kets to appear in your final expression should be \( |\psi\rangle \) and \( \langle \psi| \).

c) Compute \( \rho(t + dt) - \rho(t) \) for small \( dt \), and write the coarse-grained differential equation

\[ \frac{d}{dt} \rho(t) \approx \frac{\rho(t + dt) - \rho(t)}{dt} . \]  \hspace{1cm} (10)

Show that this has the proper Lindblad form of

\[ \frac{d\rho}{dt} = -\frac{i}{\hbar} [H_0, \rho] - \frac{1}{2} \left( C^\dagger C \rho - 2C \rho C^\dagger + \rho C^\dagger C \right) , \]  \hspace{1cm} (11)

and identify what \( C \) and \( C^\dagger \) are for this case.
8.422 Atomic and Optical Physics II
Spring 2013

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