1. This problem reviews the Boltzmann equation and compares the result with the Kubo formula. For a derivation of the Boltzmann equation, read p.319 of Ashcroft and Mermin.

(a) Consider an electron gas subject to an electron field

$$\vec{E} (\vec{r}, t) = E_0 e^{i(\vec{q} \cdot \vec{r} - i\omega t)}$$  \hspace{1cm} (1)

The Boltzmann equation in the relaxation time approximation is

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{r}} \cdot \vec{v} + \frac{\partial f}{\partial \vec{k}} (-e) \vec{E} = -\frac{f - f_0}{\tau}$$  \hspace{1cm} (2)

where $f_0$ is the equilibrium distribution

$$f_0(\epsilon) = \frac{1}{e^{\beta\epsilon} + 1}$$  \hspace{1cm} (3)

Write

$$f(\vec{r}, \vec{k}, t) = f_0(\epsilon_k) + \Phi(\vec{k}) e^{i\vec{q} \cdot \vec{r} - i\omega t}$$  \hspace{1cm} (4)

and work to first order in $\Phi(\vec{k})$ and $\vec{E}$.

Show that the conductivity is given by

$$\sigma(\vec{q}, \omega) = \frac{e^2}{4\pi^3} \int d\vec{k} \frac{\tau(\hat{e} \cdot \vec{v})^2}{1 - i\tau(\omega - \vec{q} \cdot \vec{v})} \left( -\frac{\partial f_0}{\partial \epsilon_k} \right)$$  \hspace{1cm} (5)

where $\hat{e}$ is the unit vector in the direction of $E_0$.

(b) A simple way to derive the Kubo formula is to compare the energy dissipation rate $\sigma E_0^2$ with the rate of photon absorption. At finite temperature, we need to include both absorption and emission processes. Show that for free electrons (including spin)

$$\sigma'(q, \omega) = \frac{2e^2}{m^2 \sqrt{V}} \sum_{\alpha, \beta} | < \beta | e^{i\vec{q} \cdot \vec{r}} \hat{e} \cdot \vec{p} |\alpha > |^2 \frac{(f_0(E_\alpha) - f_0(E_\beta))}{(E_\beta - E_\alpha)/\hbar} \delta (\hbar \omega - (E_\beta - E_\alpha))$$  \hspace{1cm} (6)
Using the Kramers-Kronig relation, show that the complex conductivity is

\[ \sigma(\vec{q}, \omega) = \frac{2e^2}{m^2 V} \sum_{\alpha, \beta} \left| <\beta|e^{i\vec{q}\cdot\vec{r}}\hat{e}\cdot\vec{p}|\alpha> \right|^2 \left( -i \right) \left( f_0(E_\beta) - f_0(E_\alpha) \right) \left( E_\beta - E_\alpha - \hbar \omega - i\eta \right) \]  

(7)

(c) For \(|q| \ll k_F\), show that Eq.(7) reduces to Eq.(5) under the assumptions that \(|\alpha>, |\beta>\) are plane waves and \(\eta\) is identified with \(\frac{1}{\tau}\).

2. Equation (5) in Problem 1 is valid for any relation between \(\vec{q}\) and \(\hat{e}\). In an isotropic material the response can be separated into the longitudinal (\(\vec{q} \parallel \hat{e}\)) and transverse parts (\(\vec{q} \perp \hat{e}\)). The latter is appropriate for the propagation of electromagnetic waves.

(a) For \(T \ll \epsilon_F\), show that the transverse conductivity can be written as an integration over the Fermi surface.

\[ \sigma_\perp(\vec{q}, \omega) = \frac{\sigma_0}{1 - i\omega\tau} \int_{-1}^{1} dx \frac{1 - x^2}{1 + sx} \]  

(8)

where

\[ s = \frac{iqv_F\tau}{1 - i\omega\tau} \]  

(9)

In Eq.(8) \(\sigma_0 = ue^2\tau/m\) is the DC Boltzmann conductivity and the integration variable \(x\) stands for \(\cos \theta\) in an integration over the Fermi surface.

(b) The integral in Eq.(8) can be done analytically. For our purposes, find the small \(|s|\) and large \(|s|\) limits. The small \(|s|\) limit is the Drude conductivity while the large \(|s|\) limit is called the "extreme anomalous region." It describes the situation when the electron mean free path \(\ell\) is much greater than the wavelength of light. Note that it is reduced from \(\sigma_0\) by the factor \(1/(q\ell)\). Produce a simple argument to show that this reduction factor can be understood on the basis of kinetic theory of classical particles. (\textbf{Hint:} Consider a low frequency transverse electromagnetic wave. For \(q\ell \ll 1\), all the electrons can absorb energy from the electric field. However, for \(q\ell \gg 1\), only a fraction travelling almost parallel to \(\hat{e}\) can do so. The argument was first given by Pippard.)