1. (a) We can include the effects of Coulomb repulsion by the following effective potential:

\[ V(\omega) = V_p(\omega) + V_c(\omega) \]

where \( V_p = -V_0 \) for \( |\omega| < \omega_D \) is the phonon mediated attraction and \( N(0)V_c = \mu > 0 \) for \( |\omega| < E_F \) represents the Coulomb repulsion. Write down the self-consistent gap equation at finite temperature. Show that \( \Delta(\xi) \) is frequency dependent even near \( T_c \) so that the \( T_c \) equation becomes

\[ \Delta(\xi) = -N(0) \int d\xi' V(\xi - \xi') \Delta(\xi') \frac{1 - 2f(\xi')}{2\xi'} \]  

(1)

This integral equation is difficult to solve analytically, but we may try the following approximate solution:

\[ \Delta(\omega) = \begin{cases} \Delta_1, & |\omega| < \omega_D \\ \Delta_2, & |\omega| > \omega_D \end{cases} \]

Now rewrite Eq.(1) as

\[ \Delta(\xi) = -N(0) \int d\xi' V_p(\xi' - \xi) \Delta(\xi') \frac{1 - 2f(\xi')}{2\xi'} + A \]  

(2)

where

\[ A(\xi) = -N(0) \int d\xi' V_c(\xi' - \xi) \Delta(\xi') \frac{1 - 2f(\xi')}{2\xi'} \]  

(3)

Convince yourself that \( A(\xi) \) is a slowly varying function of \( \xi \) for \( \xi < E_F \), so that we may approximate \( A(\xi) \) by \( A(0) \) in Eq.(2). Produce an argument to show that in the region \( \xi > \omega_D \) the first term in the R.H.S. of Eq.(2) is small compared with \( A \) so that in fact \( \Delta_2 \approx A(0) \). In the same spirit show that

\[ \Delta_1 \approx N(0)V_0\Delta_1 \ln \frac{\omega_D}{kT_c} + \Delta_2 \]
Combining this with an equation for $\Delta_2$ using Eq.(3), show that the $T_c$ equation becomes

\[ 1 = \ln \left( \frac{\omega_D}{kT_c} \right) (N(0)V_0 - \mu^*) \]  

(4)

where $\mu^* = \frac{\mu}{1 + \mu \ln (E_F/\omega_D)}$. $\mu^* < \mu$ is called the renormalized Coulomb repulsion. It can be thought of as an effective repulsion with a cutoff at $\omega_D$ instead of $E_F$. Equation (4) shows that the condition for superconductivity is $N(0)V_0 > \mu^*$ and not $N(0)V_0 > \mu$. For screened Coulomb repulsion, estimate $\mu$ and $\mu^*$ for a typical metal.

(b) Upon isotope substituting $M \rightarrow M + \delta M$, how is the Debye frequency affected to leading order? Assuming that this is the only effect, how is $\delta T_c/T_c$ related to $\delta M/M$, (i) in the absence of Coulomb repulsion, and (ii) including Coulomb repulsion.

2. Show that within the Heitler-London approximation for two hydrogen-like atoms located at $R_a$ and $R_b$, the singlet and triplet variational energies are given by

\[ E_{s,t} = E_a + E_b + \frac{V \pm I}{1 \pm I^2} \]

where $l = \int dr \phi_a^*(r)\phi_b(r)$ is the overlap integral,

\[ V = \int dr_1, dr_2 |\phi_a(r_1)\phi_b(r_2)|^2(\Delta H) \]

and $I$ is the exchange integral

\[ I = \int dr, dr_2, \phi_a^*(r_1)\phi_b^*(r_2)\phi_b(r_1)\phi_a(r_2)(\Delta H) \]

where

\[ \Delta H = \frac{e^2}{R_{ab}} + \frac{e^2}{r_{12}} - \frac{e^2}{r_{1b}} - \frac{e^2}{r_{2a}}. \]