Consider a Fermi gas with dispersion $\epsilon_k$ and a repulsive interaction $U\delta(r)$. Now if $N(0)U > 1$, we find in mean field theory the spontaneous appearance of the order parameter:

$$\Delta = U \langle n_\uparrow - n_\downarrow \rangle,$$

and the splitting of the up- and down-spin bands, so that the mean field Hamiltonian is

$$H_{\text{MF}} = \sum_k \left( \tilde{\epsilon}_{k\uparrow} c_{k\uparrow}^\dagger c_{k\uparrow} + \tilde{\epsilon}_{k\downarrow} c_{k\downarrow}^\dagger c_{k\downarrow} \right),$$

where

$$\tilde{\epsilon}_{k\uparrow} = \epsilon_k - \Delta/2 \quad \text{and} \quad \tilde{\epsilon}_{k\downarrow} = \epsilon_k + \Delta/2.$$

1. Consider the transverse spin susceptibility. Instead of $\chi_x = dM_x/dH_x$, where $M = -2\mu_B(\sigma/2)$ it is more convenient to consider the response to $H_+$ which couples to a spin flip excitation, i.e., $\chi_\perp = d(M_+/dH_+)$ where $M_+ = \frac{1}{2}(M_x + iM_y)$ and $H_+ = H_x + iH_y$. Show that the response function for the mean field Hamiltonian is given by $\chi_\perp^0 = \mu_B^2 \Gamma_0$ where

$$\Gamma_0(q,\omega) = \sum_k \frac{f(\tilde{\epsilon}_{k+q\downarrow}) - f(\tilde{\epsilon}_{k\uparrow})}{\omega - \tilde{\epsilon}_{k+q\downarrow} + \tilde{\epsilon}_{k\uparrow} + i\eta}.$$

This is the generalization of the Lindhard function to a spin split band.

2. Now include the interaction term in the response to the transverse field in a self consistent field approximation. Show that
\( \chi(q, \omega) = \frac{\mu^2 \Gamma_0(q, \omega)}{1 - U \Gamma_0(q, \omega)} \).

3. The poles of the numerator in \( \chi(q, \omega) \) describe the single particle-hole excitations. Sketch the region in \((\omega, q)\) space where \( \text{Im} \chi \neq 0 \) due to these excitations.

4. The other pole in \( \chi(q, \omega) \) occurs when the denominator vanishes. Calculate the dispersion of this pole which we identify as the spin wave excitation as follows:

   (a) Show that at \( q = \omega = 0 \), the denominator vanishes. [Hint: the condition \( 1 - U \Gamma_0 = 0 \) is the same as the self-consistency equation for \( \Delta(T) \).]

   (b) Expand \( \Gamma_0(q, \omega) \) for small \( q, \omega \) and show that the location of the pole of \( \chi(q, \omega) \) is given by \( \omega(q) = Cq^2 \). Note that unlike the Lindhard function for free fermions, the existence of the gap \( \Delta \) makes the expansion well behaved.