Identical particles & second quantization

For a system of \( N \) identical particles, all physical observables are invariant under an exchange of two particles.

If \( \psi(x_1, \ldots, x_N) \) is the wave function of a system of \( N \) identical particles,

\[
|\psi(x_1, \ldots, x_N)|^2 = |\psi(x_2, x_1, x_3, \ldots, x_N)|^2
\]

\[
\Rightarrow \psi(x_1, x_2, \ldots) = e^{i\theta} \psi(x_2, x_1, x_3, \ldots, x_N).
\]

As \( 2 \) exchanges return the system to its original state

\[
e^{i\theta} = 1 \Rightarrow e^{i\theta} = \pm 1.
\]
(With spin the exchange must involve spin indices too.

\[ \psi(x_1, \sigma_1, x_2, \sigma_2, \ldots) = \pm \psi(x_2, \sigma_2, x_1, \sigma_1, \ldots) \]

Symmetry under exchange imposes a restriction on the allowed Hilbert space for the system of identical particles.

For bosons \( e^{i\theta} = +1 \), and the states allowed in the Hilbert space are symmetric under exchange of any two particles – more precisely, are eigenstates of the permutation operator with eigenvalue +1.

For fermions \( e^{i\theta} = -1 \), and the states allowed in the Hilbert space are antisymmetric under exchange of any two particles – i.e., are eigenstates of the permutation operator with eigenvalue -1.
Comments:

1. Despite its familiarity, Fermi statistics is from a certain point of view quite weird. Exchange of two fermions even if they are very far away still changes the wave function (by a phase factor $-1$) can be regarded as an "infinitely" non-local "interaction" between 2 identical fermions.

2. For identical particles in 2 spatial dimensions, more exotic possibilities exist where the statistics is fractional (i.e., under exchange the phase factor $e^{i\theta \pi/2}$) or even non-abelian (where under exchange, the wave function changes by multiplication by a unitary matrix & the matrices for different exchanges do not commute). Both these exotic possibilities in d=2 are realized.
at least theoretically - in the fractional quantum Hall effect.

For the present, stick to bosons/fermions.

Given an arbitrary function $u(x_1, \ldots, x_N)$, can always symmetrize or antisymmetrize it:

$$u(x_1, \ldots, x_N) = \frac{1}{N!} \sum_P u(x_{p_1}, x_{p_2}, \ldots, x_{p_N})$$

(Bozonic)

$$= \frac{1}{N!} \sum_P \text{Sgn}(P) u(x_{p_1}, x_{p_2}, \ldots, x_{p_N})$$

(Fermionic)

where $P$ denotes a permutation of $(1, \ldots, N)$.

$\text{Sgn}(P) = +1$ for even permutation

$= -1$ "odd"
Consider a special case where

$$u(x_1, \cdots, x_N) = u_1(x_1) u_2(x_2) \cdots u_N(x_N)$$

i.e. is a product of separate functions of each of the $N$ coordinates.

Antisymmetrized wavefunction

$$\Psi(x_1, \cdots, x_N) = \frac{1}{N!} \left| \begin{array}{ccc} u_1(x_1) & u_2(x_2) & \cdots & u_1(x_N) \\ u_2(x_1) & u_2(x_2) & \cdots & u_2(x_N) \\ \vdots & \vdots & \ddots & \vdots \\ u_N(x_1) & u_N(x_2) & \cdots & u_N(x_N) \end{array} \right|$$

This is known as the Slater determinant.

Similarly the symmetrized wavefunction is

$$\Psi(x_1, \cdots, x_N) = \frac{1}{N!} \sum_{\pi} u_1(x_{\pi_1}) u_2(x_{\pi_2}) \cdots u_N(x_{\pi_N})$$

is sometimes known as the "permanent."
Note: The antisymmetric wavefunction vanishes
if \[ u_k = u_j \text{ (if } k \neq j \text{) or } x_i = x_j \text{ (if } i \neq j \text{)} \]

This is the Pauli principle for fermions.

More generally consider any complete set of orthogonalized
single particle states \(|\phi_i\rangle\).

A boson: Many particle states may be constructed as
tensor products of these single particle states
\[
|\phi_1 \ldots \phi_N\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \ldots \otimes |\phi_N\rangle.
\]

Again we may suitably symmetrize/antisymmetrize these
states as appropriate for bosons/fermions respectively.

\[
|\phi_1 \ldots \phi_N\rangle = \frac{1}{N!} \sum_{\sigma} \prod_{i=1}^{N} |\phi_{\sigma_i}\rangle
\]

\[ q = +1 \text{ for bosons} \]
\[-1 \text{ for fermions} \]
The states \( |x_1, \ldots, x_n \rangle \) form a complete orthonormal basis for the Hilbert space of the many-particle system.

**Occupation \# representation**

The information in \( |x_1, \ldots, x_n \rangle \) may also be represented by specifying the \# of times any particular \( x \) occurs in the state.

Clearly \( 0 \leq n_x < \infty \)

But for fermions since each \( x \) occurring in the state cannot be repeated, must have \( 0 \leq n_x \leq 1 \), i.e. \( n_x = 0 \) or \( 1 \) for fermions (Pauli exclusion).

For bosons \( n_x \) can be any non-negative integer a priori.

\[ \therefore \text{We may represent the state by the collection } |\xi_{n_1, \ldots, n_n} \rangle \]

of the "occupation \#" \( n_x \).
The occupation basis is clearly an equivalent complete orthonormal basis.