Vortices

Apart from the sound mode described above, there are an interesting class of excitations known as vortices, \( \Phi \) being a phase is only defined mod \( 2\pi \).

Consider a cylindrical sample with a hole in the middle.

The field \( e^{i\Phi} \) must be single-valued.

\[
\oint_C \nabla \Phi \cdot dl = 2\pi m
\]

for every curve \( C \) that winds around the hole with \( m = \text{integer} \).

States with different \( m \) are topologically distinct and cannot be continuously deformed into one another.

\( m \) counts the \# of times the phase \( \Phi \) winds around \( 2\pi \).

We can also consider such configurations directly in the bulk where \( \Phi \) winds around by some multiple of \( 2\pi \) or encircling some point - these are known as vortices.
In the core of the vortex, \( \langle \psi \rangle = 0 \) so that the phase is not well defined.

Comment - such "topological" defects are a general phenomena associated with broken symmetry states.

Rich possibilities depending on symmetry of order parameter space & dimensionality of coordinate space.

For a vortex with \( m = 1 \),

\[
\Delta \phi \sim \frac{m}{r}
\]

\[
E \sim L \int d\theta dr r \frac{f_s(m)}{2} (\frac{m}{r})^2
\]

\[
\alpha \frac{\pi L f_s m^2 \ln \frac{L}{a}}{a} \quad \text{vortex "core" size}
\]

Note that once the U(1) symmetry is broken, we can no longer label the excitations of the system by \( \psi \) (it loses its integrity as a good quantum number).
However, there emerges a new (topological) quantum number - the vorticity - which is conserved by the dynamics.

We may label the states in the Hilbert space by their total vorticity.

To summarize, there are two distinct classes of excitations in a neutral superfluid.

- The sound mode & vortices

  Sound mode is gapless with \( k \sim |k| \) & controls low-\( T \) thermodynamics.

  Vortices are topologically stable objects which interact with each other through long ranged forces mediated by the sound modes.

  (Divergence of energy of a single vortex is also due to its interaction with the sound mode.)
Superfluid equations of motion

The "hydrodynamic" variables that are necessary to describe the long length scale & long time physics of the superfluid state at zero temperature are the phase $\phi$ of the order parameter $\psi(x)$ & the local particle density $\rho(x)$. (See previous lectures)

As hinted by action on page 71, these are canonically conjugate to each other.

To see this formally, consider

\[
\left[ \psi(x), \psi^\dagger(x') \right] = \delta(x-x')
\]

\[
\Rightarrow \left[ \psi(x), \psi^\dagger(x') \psi(x') \right] = \psi(x) \delta(x-x')
\]

For states that are smooth long distance deformations of the superfluid ground state, write

\[
\psi(x) \approx \psi_0 e^{i \phi(x)}
\]

so that
More careful discussion

Actually must write \( \hat{\psi}(x) = \sqrt{P(x)} \ e^{i \phi(x)} \)

with \( P(x) = P_0 + SF(x) \)

In the relevant portion of the Hilbert space, matrix elements of \( SF(x) \) will have small absolute values compared to \( P_0 \)

\( \hat{G}^+(x) \ \hat{\psi}(x) = \left( e^{-i \phi(x)} \ \sqrt{P(x)} \right) \left( \sqrt{P(x)} \ e^{i \phi(x)} \right) \approx P(x) \)

\[ \left[ \sqrt{P_0 + SF} \ e^{i \phi(x)} \ P(x') \right] = \sqrt{P_0 + SF} \ e^{i \phi(x')} \]

Now approximate \( \sqrt{P_0 + SF} \approx \sqrt{P_0} \) in the factor in front of \( e^{i \phi(x)} \) to get

\[ \left[ e^{i \phi(x)} \ P(x') \right] = e^{i \phi(x')} P(x-x') \]
\[ \left[ e^{i \hat{\Theta}(x)}, \hat{P}(x') \right] = e^{i \Theta(x)} \delta(x - x') \]

This commutation relation is satisfied if
\[ \left[ \hat{\Theta}(x), \hat{P}(x') \right] = -i \delta(x - x') \]

(As a check, consider expansion of 1st eqn above in powers of \( \hat{\Theta} \) & keep leading non-vanishing term in both LHS & RHS)

More formal proof: Start with \( [e^{i \hat{\Theta}}, \hat{P}] = e^{i \Theta} \delta(x - x') \)

Differentiate both sides w.r.t \( \Theta \)

Start with \( [\hat{\Theta}(x), \hat{P}(x')] = -i \delta(x - x') \)

\[ \Rightarrow \text{can write} \quad \hat{P}(x') = i S \quad \delta(x') \]

\[ \Rightarrow \quad \left[ e^{i \hat{\Theta}(x)}, \hat{P}(x') \right] = i \left( e^{i \Theta(x)} S - S e^{-i \Theta(x)} \right) \]

\[ = e^{i \Theta(x)} \delta(x - x') \quad \text{as required} \]
Thus $\phi$ and $\tilde{\phi}$ are conjugate to each other.

Now consider that $H$ is a generic Hamiltonian for the interacting Bose system.

Again restricting to states which are smooth long distance deformations of the superfluid ground state, we may write

$$H = \int \left[ \frac{\hbar^2}{2m} \hat{\rho}(x)^2 - \hbar \int \rho(x) \hat{\Phi}(x) \right]$$

where $\hat{\rho}(x) = \text{local density}$

$\hat{\Phi}(x) = \text{local phase of order parameter}$

$H = \text{Hamiltonian density}$

$$\frac{\partial \hat{\rho}(x)}{\partial t} = -\left\langle \frac{\delta H}{\delta \hat{\Phi}(x)} \right\rangle = -\mu(x,t)$$

$\mu(x,t) = \text{local chemical potential}$

The density satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j}$$
From the microscopic expression for the current operator

\[ \vec{j} = \frac{\hat{\psi}^+ (-i \vec{v}) \hat{\psi} + h.c}{2m} \]

and letting \( \hat{\psi} = \psi_0 e^{i\theta} \),

we get \( \vec{j} = \frac{\psi_0^2 \vec{v} \theta}{2m} = \frac{\vec{J}_0 \theta}{2m} \)

(More generally, we should allow for some arbitrary coefficient \( \theta_0 \) instead of \( \theta_0 \)).

Then \( \frac{\partial \vec{J}_0}{\partial t} = -\frac{\vec{J}_0}{2m} \vec{v}^2 \theta \)

Note that \( \frac{\partial \vec{v} \theta}{\partial t} = -\vec{v} \mu \)

⇒ If \( \theta \) is uniform throughout the sample, the system cannot sustain chemical potential differences

\[ \Rightarrow \frac{\partial \vec{j}}{\partial t} = -\frac{\vec{J}_0}{2m} \vec{v} \mu \]
If there is an applied chemical potential difference, the current will increase.

In particular if \( \frac{\partial \mu}{\partial x} = 0 \), \( \frac{\partial j}{\partial x} = 0 \), i.e. a superflow.

How can the current degrade?

This requires the motion of vortex lines.

Consider a 2-d section

\[
\begin{array}{c}
\theta_2 \\
\theta_1
\end{array}
\]

Whenever a vortex moves from left to right, \( \theta_2 \) winds by \( 2\pi \) relative to \( \theta_1 \).

\( \Rightarrow \) If \( \dot{\eta} \) is rate at which vortices move from left to right,

\[
\dot{\theta}_2 - \dot{\theta}_1 = 2\pi \dot{\eta} = -(\mu_2 - \mu_1)
\]

\( \Rightarrow \) Vortex motion can produce a chemical potential difference.
For more insight, consider a toroidal ring.

In the ground state $\phi = \text{uniform}$

$$\uparrow \quad \uparrow \quad \uparrow \quad \Rightarrow \quad \text{no current}.$$  

Now consider a state where $\phi$ winds by $2\pi n$ around the hole $\Rightarrow$ there are $n$ vortices trapped in the hole.

$$\uparrow \quad \uparrow \quad \Rightarrow \quad \phi \text{ is non-uniform}$$  

$$(n=1)$$  

$$\Rightarrow \quad \text{there is a circulating current}.$$  

Degrading the current to return system to ground state requires the trapped vortices to escape from the hole.

- This has a huge energy barrier $\Rightarrow$ very improbable for a large system.

$\therefore$ Topological stability of vortex guarantees persistence of superflow.
Revisit the sound (i.e. Goldstone) mode from the equs of motion:

\[ \frac{\partial^2 \phi}{\partial t^2} = -\frac{\mu}{\kappa} \quad ; \quad \frac{\partial \phi}{\partial t} = -\frac{\phi}{\kappa} \frac{\nabla^2 \phi}{m} \]

\[ \frac{\partial^2 \phi}{\partial t^2} = -\frac{\partial \phi}{\partial t} \frac{\nabla^2 \phi}{m} \]

With short-ranged interactions, change in local value of \( \mu \) is related to local change in density

\[ 8\mu = \frac{\partial \phi}{\partial t} \]

where \( \kappa \) = compressibility of the system.

( Assume no externally imposed chemical potential gradients)

\[ \frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{\kappa} \frac{\partial \phi}{\partial t} = \frac{\rho}{m} \frac{\nabla^2 \phi}{m} \]

This describes a sound wave with \( \omega = c k \)

with \( c^2 = \frac{\rho}{m} \frac{1}{\kappa} \)

Alternatively, can eliminate \( \phi \) to get an eqn. for \( n \):

\[ \frac{\partial^2 n}{\partial t^2} = \frac{\rho}{m} \frac{\partial^2 \phi}{\partial t^2} = \frac{\rho}{m} \frac{\nabla^2 \phi}{m} \]
Thus the sound wave may be thought of as an oscillation in the density of the liquid.

**Charged bosons**

What happens if the bosons that undergo the superfluid transition are charged?

This is the case with superconductivity in metals which can be thought of as the superfluidity of Cooper pairs which are charged bosons.

The sound wave mode being an oscillation of the charge density will be costless.

[Change in density in one part of the sample creates a long ranged electric static potential that affects the behavior in distant parts.]

Consider the signs of motion

\[
\frac{d\phi}{dt} = -\frac{1}{\mu}, \quad \frac{d\Sigma}{dt} = -\frac{\phi}{\mu} \frac{d\phi}{dt}
\]
Unlike with short-ranged interactions,
changes in $\mu$ are not determined by the local
change in density.

To handle this situation, divide $\mu$ into two parts

$$\mu = \mu_{\text{loc}} + eV$$

where $\mu_{\text{loc}}$ is the electrostatic potential
determined by the charge density in distant parts
of the sample, then the Gauss law

$$\nabla^2 V = -4\pi e\rho$$

Changes in $\mu_{\text{loc}}$ are determined by the local change
in density as the effect of the long-range interaction
have been subtracted out.

The $\mu$ and become

$$\frac{d\mu}{dt} = -\left(\mu_{\text{loc}} + eV\right)$$

$$\frac{d\phi}{dt} = -\frac{\rho}{m} \nabla^2 \phi$$

$$\nabla^2 V = -4\pi e\rho$$
\[ \frac{\partial^2 \phi}{\partial t^2} - \frac{1}{m} \nabla^2 \phi = \pm \frac{\phi}{\sqrt{m} \tau} \left( \nabla \phi \cdot \nabla \phi + \epsilon \frac{\nabla^2 \phi}{\sqrt{m} \tau} \right) \]

\[ = \pm \frac{\phi}{\sqrt{m} \tau} \left( \frac{\nabla \phi \cdot \nabla \phi}{m} + \epsilon \frac{\nabla^2 \phi}{m} \right) \]

\[ = \pm \frac{\phi}{\sqrt{m} \tau} \left( \frac{\nabla \phi \cdot \nabla \phi}{m} + \frac{\nabla^2 \phi}{m} \right) \]

\[ \Rightarrow \text{Dispersion} \quad \omega^2 = \frac{\phi}{\sqrt{m} \tau} \left( \frac{k^2}{m} + \frac{\nabla^2}{m} \right) \]

\[ = \left( \text{"plasma frequency"} \right) \]

\[ \Rightarrow \quad \omega \sim \omega_0 \quad \text{as} \quad k \to 0. \]

Thus the sound mode has become gapped.

- in this context, it is known as a plasmon

(see plasma oscillations)

This result illustrates an important exception to Goldstone's theorem - if there are long ranged forces, then the Goldstone mode may become gapped (the Anderson-Higgs phenomenon).

E.g.: In systems where the symmetry being "broken" is a gauge symmetry (like in a SC), the Goldstone mode is gapped.
These ideas are at the heart of application of broken symmetry concepts to the standard model.

\( W^\pm, Z \) particles are the analogs of the plasma.

Now return to the neutral superfluid

\[
[\mathcal{O}(x), \mathcal{S}(x')] = -i \mathcal{S}(x-x')
\]

implies suggests that \( \mathcal{O} \) & \( \mathcal{S} \) cannot simultaneously be specified to arbitrary accuracy.

In particular if we choose as a ground state, a state with \( \mathcal{O} \) uniform & fixed then it suggests that such a state will not have definite particle #.

To see this explicitly, note that for a state with fixed particle # \( N \)

\[
\langle \phi_N | \hat{\psi} | \phi_N \rangle = 0 \quad \text{as} \quad \hat{\psi} \text{ only has non-zero matrix elements between states with different } N,
\]

\( \Rightarrow \) for a closed system with \( N \) bosons, cannot have \( \langle \psi \rangle = 0 \) strictly.
What then are we to make of our discussion of superfluidity, which assumed \( \Psi \neq 0 \)?

To see what is going on, consider a large system with \( N \) particles in a closed box of volume \( V \).

Consider the path integral

\[
Z = \int \mathcal{D}\Psi e^{iS}
\]

\[
S = \int d^d \mathbf{x} \left( \Psi^* \frac{\partial S}{\partial \Psi} \Psi + \frac{\Psi^*}{2m} \frac{\partial^2 S}{\partial \mathbf{p}^2} \Psi + \mu |\Psi|^2 - \mu' |\Psi|^4 \right)
\]

As before, proceed as before. Minimizing classical action to get \( \Psi = \Psi_0 = \phi \), then write \( \Psi = (\phi \phi^* \phi^0) e^{i\phi} \).

In the earlier derivation we expanded to 2nd order in \( \phi^0 \) and integrated it out to get

\[
S_{\text{eff}}[\phi] = \int d^d x \frac{\hbar^2}{2m} \left( \frac{\partial \phi^*}{\partial \phi} \frac{\partial \phi}{\partial \phi^*} - \frac{1}{2m} (\nabla \phi)^2 \right).
\]

To study the effects of the fluctuations...
In the earlier derivation, we expanded to 2nd order in $s$ to obtain

$$S[s, 0] = \int dx \frac{d}{dt} \left( -s \frac{\partial V}{\partial s} - \frac{1}{2} \frac{\partial}{\partial s} (s^2) \right) + \frac{(s^2)}{2K}$$

To study the fluctuations of the total particle $\# 2$ and its conjugate, consider only uniform configuration $\phi, \phi^* \phi = \phi^0 \phi^0$, $s_2 = s_2^0$ independent of $x$.

$$S[s_2^0, 0] = \sqrt{\sum} \int dt \left[ -s_2^0 \frac{d\phi^0}{dt} - \frac{1}{2} \frac{\partial (s^2)}{\partial s} \right]$$

Introduce $sN = \sqrt{s_2^0} = \text{change in total particle } \#$

$$S[SN, 0] = \int dt \left[ -SN \frac{d\phi^0}{dt} - \frac{(SN)^2}{2VK} \right]$$

$$\Rightarrow SN_0 \neq \phi^0 \text{ are canonically conjugate}$$

$$[\phi^0, SN_0] = -i \text{ in quantum formulation}$$
Can now integrate $SN_0$ to get

$$S_{\text{eff}} \{ \theta_0 \} = \frac{1}{2} \frac{V}{\kappa} \int dt \, (\theta_0)'^2$$

which is action for QM particle on a ring

of unit radius

$\theta_0$ : angular coordinate of ring

$\kappa V$ : role of "mass"

Equivalently can write Hamiltonian

$$H_{\text{eff}} = \frac{1}{2V} (SN)^2 - \kappa \left( N-N_0 \right)^2$$

Key point: "Mass" of particle $\propto V \rightarrow \infty$ in thermodynamic limit $\Rightarrow$ can treat particle "classically"

$\Rightarrow$ can ignore non-commutativity of $\theta_0 \neq SN_0$

$\Rightarrow$ can assign simultaneous eigenvalues $\leq \theta_0 \leq SN_0$

Alternate: if $N_0$ is particle $\#_0$ in ground state,

there exist excited states with $N-N_0 \sim O(1)$ with energy...
\[ \text{In the thermodynamic limit, all these states become degenerate.} \]

(Note that bound wave states have energy \( o(1) \)

\[ \Rightarrow \chi \text{ for } d \gg 1. \]

Equivalently for a large system if we initially prepare the system in a "wave packet" with reasonably well defined \( \Theta_0 \), it will evolve the uncertainty in \( \Theta_0 \) will evolve very slowly.)