1. **Operator identities.**

Here we prove two useful theorems from operator algebra that will be used in the problems of this homework and later in the course.

a) Let \( \hat{A} \) and \( \hat{B} \) be two operators that do not necessarily commute. Prove the so-called **operator expansion theorem**:

\[
f(x) = \exp(x\hat{A}) \hat{B} \exp(-x\hat{A}) = \hat{B} + x[\hat{A}, \hat{B}] + \frac{x^2}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \ldots
\]

with \( x \) a parameter. (Hint: consider the derivative \( f'(x) \) and compare its Taylor series in \( x \) with that for \( f(x) \).)

In the special case when the commutator \( [\hat{A}, \hat{B}] = c \) is a c-number, the series terminates after the second term, giving

\[
\exp(x\hat{A}) \hat{B} \exp(-x\hat{A}) = \hat{B} + cx
\]

Apply this result to the coordinate and momentum operators, \( \hat{B} = \hat{q}, \hat{A} = \hat{p}/\hbar = -i\hbar/d\hat{q} \).

b) Let \( A \) and \( B \) be two operators whose commutator \( [\hat{A}, \hat{B}] \) commutes with both \( A \) and \( B \) (e.g., \( [\hat{A}, \hat{B}] = c \) a c-number). Prove the **Campbell-Baker-Hausdorff theorem**:

\[
\exp [x(\hat{A} + \hat{B})] = \exp(x\hat{A}) \exp(x\hat{B}) \exp(-x^2[\hat{A}, \hat{B}]/2)
\]

For that, consider \( \hat{C}(x) = \exp(x\hat{A}) \exp(x\hat{B}) \), differentiate both sides with respect to \( x \) and, using the operator expansion theorem (1), show that \( d\hat{C}(x)/dx = (\hat{A} + \hat{B} + x[\hat{A}, \hat{B}]) \hat{C}(x) \). Integrate with respect to \( x \) like an ordinary differential equation.

2. **Displacement operators.**

a) Consider the displacement operators, defined as

\[
\hat{D}(v) = \exp\left(v\hat{a}^+ - \bar{v}\hat{a}\right)
\]

with \( v \) a complex parameter. Prove unitarity: \( \hat{D}^+(v) \hat{D}(v) = 1, \hat{D}^{-1}(v) = \hat{D}(-v) \).

For a real-valued \( v \) show that in the \( q \)-representation the displacement operator (4) acts as an argument shift:

\[
\hat{D}(v) \psi(q) = \exp(v\partial_q) \psi(q) = \psi(q + \bar{v}), \quad \bar{v} = \sqrt{2\lambda v}
\]

with the length \( \lambda = \sqrt{\hbar/m\omega} \). (Hint: relate \( \hat{D}(v) \) to the Taylor series formula.)

b) Show that the coherent states can be obtained by “displacing” the vacuum state, \( |\psi\rangle = \hat{D}(v)|0\rangle \). (Use the operator expansion theorem (1)).

c) Show that the unitary transformation \( \hat{D}(v) \) displaces \( \hat{a} \) by \( v \), and \( \hat{a}^+ \) by \( \bar{v} \),

\[
\hat{D}^+(v) \hat{a} \hat{D}(v) = \hat{a} + v, \quad \hat{D}^+(v) \hat{a}^+ \hat{D}(v) = \hat{a}^+ + \bar{v}
\]
For any function of operators $\hat{a}$ and $\hat{a}^+$ with a power series expansion, show that

$$\hat{D}^+(v) f(\hat{a}, a^+) \hat{D}(v) = f(\hat{a} + v, a^+ + \nu)$$

(7)

d) Prove the product formula

$$\hat{D}(v') \hat{D}(v) = e^{v' \sigma - \sigma' v} \hat{D}(v + v')$$

(8)

Note that the displacement operators $\hat{D}(v)$ and $\hat{D}(v')$ commute only when $\text{arg}(v) = \text{arg}(v')$.

3. **Harmonic oscillator excited by an external force.**

   a) Consider a particle moving in a parabolic potential in the presence of a time-dependent force, $\mathcal{H} = \frac{1}{2} \hbar \omega (\hat{p}^2 + \hat{q}^2) - F(t) \hat{q}$. Show that the evolution in time of an arbitrary coherent state can be obtained using the displacement operators (4) studied in Problem 2. Assume that the evolved coherent state remains a coherent state at all times, so that

$$|\alpha \rangle(t) = \hat{D}(v(t)) |\alpha \rangle = |\alpha + v(t)\rangle$$

(9)

Obtain a differential equation for the function $v(t)$ and show that its real and imaginary parts correspond to the classical Hamilton equations $dq/dt = p$, $dp/dt = F(t)$.

Show that the unitary transformation $\mathcal{H}' = \hat{D}(v(t)) \mathcal{H} \hat{D}^{-1}(v(t))$ gives a free oscillator Hamiltonian with $F(t) = 0$. It describes the transformation of the quantum problem to the classical co-moving reference frame.

How does the function $v(t)$ should evolve in time in order for is such that at all times it remains a coherent state

b) The harmonic oscillator of part a), initially in the ground state, was subject to a constant force during the time interval $0 < t < \tau$. Find the state at $t > \tau$. Determine the distribution of energies.

c) For the state found in part b) at $t > \tau$, find the phase-space density, i.e., the Wigner function $W(q,p)$, as a function of time.