VII Biological Oscillators

During class we consider the following two coupled differential equations:

\[
\begin{align*}
\dot{x} &= -x + ay + x^2y \\
\dot{y} &= b - ay - x^2y
\end{align*}
\]  

[VII.1]

From the phase plane analysis (see L9_notes.pdf) it was clear that for certain values of a and b this system exhibits periodic oscillations as a function of time. Let us analyze [VII.1] in more detail. The nullclines are:

\[
\begin{align*}
y &= \frac{x}{a + x^2} \\
y &= \frac{b}{a + x^2}
\end{align*}
\]  

[VII.2]

There is only one fixed point \((x^*, y^*)\):

\[
\begin{align*}
x^* &= b \\
y^* &= \frac{b}{a + b^2}
\end{align*}
\]  

[VII.3]

The matrix \(A\) is (using [V.4] and [V.5]):

\[
A = \begin{bmatrix}
-1 + 2x^*y^* & a + (x^*)^2 \\
-2x^*y^* & -(a + (x^*)^2)
\end{bmatrix}
\]  

[VII.4]

The determinant and trace are:

\[
\Delta = a + b^2 > 0 \\
\tau = \frac{-b^4 + (2a - 1)b^2 + (a + a^2)}{a + b^2}
\]  

[VII.5]

The fixed point is stable when \(\tau < 0\). The region in a-b-parameter space where the system is oscillating (stable limit cycle) and is not oscillating (stable fixed point) is illustrated in Fig. 10.
Figure 11. a-b-parameter space indicating for which values of a and b the system exhibits stable oscillations and a stable fixed point

MATLAB code 5: Limit cycle

% filename: cyclefunc.m
function dydt = f(t,y,flag,a,b)
dydt = [-y(1)+a*y(2)+y(1)*y(1)*y(2);
        b-a*y(2)-y(1)*y(1)*y(2)];
plot(y(1),y(2),'.');
drawnow;
hold on;
axis([0 2 0 2]);

% filename: limitcycle.m
close;
clear;
a=0.1;
b=0.5;
options=[];
[t y]=ode23('cyclefunc',[0 50],[0.6 1.4],options,a,b);
plot(y(:,1),y(:,2));