For massive fermions,

\[ E \chi_- = -\frac{\mathbf{\sigma} \cdot \mathbf{p}}{m} \chi_- - m \chi_+ \]
\[ E \chi_+ = \frac{\mathbf{\sigma} \cdot \mathbf{p}}{m} \chi_+ - m \chi_- \]

or

\[ \mathbf{\sigma} \cdot \mathbf{p} \chi_- = -\chi_- - \frac{m}{E} \chi_+ \]
\[ \mathbf{\sigma} \cdot \mathbf{p} \chi_+ = \chi_+ + \frac{m}{E} \chi_- \]

from which we find the two helicity eigenstates

\[ \chi_- + \frac{m}{2E} \chi_+ + \frac{m}{2E} \mathbf{\sigma} \cdot \mathbf{p} \chi_- = \chi_- \]

Note: the "wrong-sign" helicity is suppressed by a factor \( m/E \); Gauge fields preserve fermion chiralities in their couplings.

\[ \mathbf{\sigma} \cdot \mathbf{p} \left( \chi_+ + \frac{m}{2E} \chi_- \right) = \left( \chi_+ + \frac{m}{2E} \chi_- \right) \]
Transformation

\[ \tilde{\psi} \left( i \gamma^\mu \gamma_5 \partial_\mu - m \right) \psi = 0 \]

\[ \left[ i \frac{\partial^2}{\partial x^\mu} - m \right] \psi(x) = 0 \quad \Rightarrow \quad \psi(x') = \Lambda(x) \psi(x) \quad \Rightarrow \quad \psi' = \Lambda \psi \]

\[ \left[ i \gamma^\mu \frac{\partial}{\partial x^\mu} - m \right] \psi(x) = 0 \quad \Rightarrow \quad \psi'(x') = \Gamma(x') \psi(x) \quad \Rightarrow \quad \psi' = \Gamma \psi \]

\[ \left[ i \gamma^\mu \gamma_5 \frac{\partial}{\partial x^\mu} - m \right] \psi(x) = 0 \quad \Rightarrow \quad \psi'(x') = \Gamma_{\mu \nu} \gamma_5 \psi(x) \quad \Rightarrow \quad \psi' = \Gamma \psi \]

\[ \Gamma_{\mu \nu} = \gamma_5 \gamma^\mu \gamma_5 \gamma^\nu \]

Thus, commute with $\Lambda$.

Also, $\psi' = \tilde{\psi} \gamma_5$.

Thus, commute with $\Lambda$.

Since $\gamma^\mu$ is not changed. \[ \left\{ \gamma^\mu, \gamma^\nu \right\} = 2 g^{\mu \nu} \]

\[ \bar{\psi} ' \gamma^\nu ' \psi = \bar{\psi} s^\dagger s \gamma^\nu \psi = \bar{\psi} \gamma^\nu \psi \quad \text{scalar} \]

\[ \bar{\psi} ' \gamma^\mu ' \psi = \bar{\psi} s^\dagger s \gamma^\mu \psi = \gamma \bar{\psi} \gamma^\mu \psi \quad \text{vector} \]

\[ \bar{\psi} s^\dagger s \gamma^\mu ' \psi = \bar{\psi} s^\dagger s \gamma^\mu \psi = -\gamma \bar{\psi} s^\dagger s \gamma^\mu \psi \quad \text{Axial vector} \]

\[ \psi ' = \tilde{\psi} \gamma_5 s \gamma_5 \psi \]

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\[ \gamma^\mu \frac{\partial \psi(x)}{\partial x^\mu} - m \psi(x) = 0 \]
\[ \psi'(x') = S \psi(x) \]
\[ i \gamma^\mu \frac{\partial \psi(x)}{\partial x'^\mu} - m \psi(x') = 0 \]
\[ S^{-1} \gamma^\mu S = \Lambda \gamma^\mu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \gamma^\mu \]
\[ S = \gamma^0 \cdot \Lambda \gamma^0 \]
\[ \Psi_{1,2} = \Psi_{1,2} \quad \Psi_{3,4} = -\Psi_{3,4} \]
\[ P \phi e^- = - P \phi e^+ \]
\[ P = (\cos \frac{\pi}{2} )^{l+1} \quad \gamma_0 = (\cos \frac{\pi}{2} )^{l+s} \]
\[ \partial_x p \phi(x) = 0 \]
Alternative method: Parity

\[ t \to t, \quad \vec{x} \to -\vec{x}, \quad \psi(t, \vec{x}) \to \psi^p(t, -\vec{x}) \]

\[ \left( i \gamma^0 \frac{\partial}{\partial t} + i \gamma^0 \gamma^5 \cdot \vec{\gamma} \right) \psi^p(t, -\vec{x}) - m \psi^p(t, -\vec{x}) = 0 \]

Since \( \gamma^0 \gamma^5 = -\gamma^5 \), so

\[ (i \gamma^0 \partial_t + i \gamma^0 \gamma^5 \cdot \vec{\gamma}) \psi^p(t, -\vec{x}) - m \psi^p(t, -\vec{x}) = 0 \]

Thus \( \gamma^0 \psi^p(t, -\vec{x}) = \psi(t, \vec{x}) \)

or \( \psi^p(t, -\vec{x}) = \gamma^0 \psi(t, \vec{x}) \)

Note in \( \gamma^5 \)-diagonal representation \( \gamma^5 \psi(t, \vec{x}) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \psi(t, \vec{x}) \)

\[ \psi^p(t, -\vec{x}) = \begin{pmatrix} \chi_- \\ \bar{\chi}_+ \end{pmatrix} e^{-ip_m x^\mu} \]

i.e. interchange of L-H and R-H chiralities!! In particular, a parity invariance theory must have both RH & LH chiralities equally!! Weak Interaction has only LH, so violate parity max.

In \( \gamma^5 \)-diagonal scheme,

\[ \gamma^5 \psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi \quad \psi = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \]

So parity = 1 for \( u_{1,2} \ E > 0 \) particles

\[ = -1 \quad u_{3,4} \ E < 0 \] anti-
Gauge Theories

General Relativity
(Theory of gravitation)

Electroweak Theory
(QED + Weak Interactions)

Quantum Chromodynamics
(Strong interactions between quarks)

All of them based on the principle of
Local Gauge Invariance

All forces are carried by
massless “bosons”
(gravitons, photons, gluons)

with the exception of the
weak interactions
based on the principle of

“Spontaneously Broken”
Local Gauge Invariance
Gauge theory of electrons interacting with photons (quanta of the electromagnetic field).

In a theory with only electrons the quantum-mechanical description of an electron involves a complex field \( \Psi(x) \)

\[
x: x^\mu = (t, \vec{x}); \mu = 0, 1, 2, 3
\]

Probability of finding an electron at "point" \( x_\mu \)

\[
|\Psi(x)|^2 = \overline{\Psi}(x) \Psi(x)
\]
\[ M_0 + M_v \]

\[ (\mu \rightarrow e\nu\nu\bar{\nu}, E_\nu > 10\text{MeV}) = 1.2 \times 10^{-2} \rightarrow 1.3 \pm 0.1 \times 10^{-2} \]

\[ M_0^2 \rightarrow \text{BR.} \]
\[ \frac{1}{2} M_0^2 \text{soft } \ll \frac{M_0}{M_\nu} \text{ hard } \]
\[ E_\nu, m_\nu = 0 \]

\[ + M_3^2 + \frac{R_0}{M_0^2 M_\nu^2} + \frac{M_\nu^2}{2} \]  
\[ \alpha^2 \ 	ext{ finite} \]

\[ \Sigma = \text{finite} = 1 + \left(\frac{\alpha}{2\pi}\right) (\pi^2 - 25/6) \]
\[ = 1 + 4.2 \times 10^{-3} \]
EM \ U(1) local Gauge transformation
\[ \psi(x) \rightarrow e^{i \lambda(x)} \psi(x), \quad \bar{\psi}(x) \rightarrow e^{-i \lambda(x)} \bar{\psi}(x) \]

\( \lambda \) continuous real \# : Abelian gp \( U(1) \)
\[ U(\lambda_1) U(\lambda_2) = U(\lambda_2) U(\lambda_1) \]
\[ (d_{\alpha x} - ie A_{\mu}) \]
\( (\not \! p - m) \psi = 0 \rightarrow (\frac{d}{d x^\mu} - ie A_{\mu}) \psi = D_{\mu} \psi = 0 \)
\[ \psi = -e^{m \not \! p} e^{-i \lambda} \]

Free e without field is not Gauge Invariant!
\[ \partial_{\mu} \psi \rightarrow e^{i \lambda(x)} \partial_{\mu} \psi + ie \psi \partial_{\mu} \lambda \]

Define covariant momentum include the field \( A_{\mu} \)
\[ D_{\mu} \psi \rightarrow e^{i \lambda} D_{\mu} \psi, \quad D_{\mu} = \partial_{\mu} - ie A_{\mu} \]

where \( A_{\mu} \rightarrow A_{\mu} + \frac{1}{e} \partial_{\mu} \lambda \)
\[ \mathcal{L} = i \bar{\psi} \gamma^\mu D_{\mu} \psi - m \bar{\psi} \psi \]
\[ = \bar{\psi} (i \gamma^\mu \partial_{\mu} - m) \psi + e \bar{\psi} \gamma^\mu A_{\mu} \psi \]

Gauge invariant!

Potential term.
Charge conjugation: $e^- \rightarrow e^+$
\[ t^c \rightarrow t \]
\[ X^c \rightarrow X \]

$e^-$ with charge $g = -e$

$\psi = (p - m) u(p) \Rightarrow \left[ \gamma^\mu \left( i \frac{\partial}{\partial x^\mu} + e A_\mu \right) - m \right] \psi = 0 \quad - (1)$

$e^+ \quad g = +e$

\[ \left[ \gamma^\mu \left( i \frac{\partial}{\partial x^\mu} \mp e A_\mu \right) - m \right] \psi_c = 0 \quad - (2) \]

**How to relate $\psi_c$ with $\psi$?**

1. $\left[ -i \gamma^\mu (i \partial_\mu - e A_\mu) - m \right] \psi^* = 0 \quad \gamma^2 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$

   \[ \delta_{i,3} = \delta_{i,3} \]

   Move $\gamma^2$ to left

   \[ \gamma^2 \left[ \gamma^\mu (1 \partial_\mu - e A_\mu) - m \right] \gamma^2 \psi^* = 0 \quad - (3) \]

   Compare (2) & (3) \[ \psi_c = i \gamma^2 \psi^* (t, x) \]

   L-phase

\[ \psi^* = \left( \begin{array}{c} 0 \\ -i \sigma_2 \end{array} \right) \left( \begin{array}{c} \psi_c^* \\ \psi \end{array} \right) = \left( \begin{array}{c} i \sigma_2 \psi_c^* \\ -i \sigma_2 \psi \end{array} \right) \]

\[ \left( \begin{array}{c} \psi_R \\ \psi_L \end{array} \right) = \left( \begin{array}{c} \psi_c \\ \psi \end{array} \right) = \left( \begin{array}{c} \psi \end{array} \right) \}

Weaker Interaction violates C maximally.
CP Transformation

\[
\Psi_L^c = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \Psi_L^c = \begin{pmatrix} 0 & 0 \\ i\sigma^2 \chi_-^* \\ 0 \\ 0 \end{pmatrix}
\]

\[
\therefore (\Psi_L^c)^c_P = \sigma^c (\Psi_L^c)^c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ i\sigma^2 \chi_-^* \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -i\sigma^2 \chi_-^* \end{pmatrix}
\]

i.e. \( (\Psi_L^c)^c_P = (\Psi^c_P)_L \)

\[\Psi_L = \Psi_R \]

\[\bar{\Psi}_L = \bar{\Psi}_R \]

Chiral Parity

Having only L-H spinor can be CP invariant, but not separately C or P invariant!!

Now \( m_Y \neq 0 \)

- Majorana spinor: \( \Psi_M^c = \Psi_M \)

i.e.

\[
\begin{pmatrix} \chi_+^* \\ \chi_- \end{pmatrix} = i\sigma^2 \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \chi_+^* \\ \chi_- \end{pmatrix}
\]

\[
\therefore i\sigma^2 \chi_-^* = \chi_+^* \quad , \quad -i\sigma^2 \chi_+^* = \chi_-
\]

e.g. given \( \chi_+^* \), \( \Psi_R = (\chi_+^*) \) and construct

\[
\Psi_M = \begin{pmatrix} \chi_+^* \\ -i\sigma^2 \chi_+^* \end{pmatrix} = \Psi_R + (\Psi_R)^c = \Psi_M^c
\]

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