8.821 String Theory
Fall 2008

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In today’s lecture we will talk about:

1. AdS wave equation near the boundary.
2. Masses and operator dimensions: \( \Delta(\Delta - D) = m^2 L^2 \).

**Erratum:** The massive geodesic equation \( \ddot{x} + \Gamma \dot{x} \ddot{x} = 0 \) assumes that the dot differentiates with respect to proper time.

Recap: Consider a scalar in \( \text{AdS}_{p+2} \) (where \( p + 1 \) is the number of spacetime dimensions that the field theory lives in). Let the metric be:

\[
d s^2 = L^2 \frac{d z^2 + d x^\mu d x_\mu}{z^2},
\]

then the action takes the form:

\[
S[\phi] = -\frac{\kappa}{2} \int d^{p+1}x \sqrt{g} \left( (\partial \phi)^2 + m^2 \phi^2 + b \phi^3 + ... \right),
\]

where \( (\partial \phi)^2 \equiv g^{AB} \partial_A \phi \partial_B \phi \) and \( x^A = (z, x^\mu) \). Our goal is to evaluate:

\[
\ln(\exp - \int d^{p+1}x \phi^O)_{\text{CFT}} = \text{extremum}_{[\phi \rightarrow \phi_0 \text{ at } z = \epsilon]} S[\phi],
\]

where \( S[\phi] \equiv S[\phi^*(\phi_0)] \equiv W[\phi_0] \), i.e. by using the solution to the equation of motion subject to boundary conditions. Now Taylor expand:

\[
W[\phi_0] = W[0] + \int d^Dx \phi_0(x) G_1(x) + \frac{1}{2} \int \int d^Dx_1 d^Dx_2 \phi_0(x_1) \phi_0(x_2) G_2(x_1, x_2) + ...
\]

where

\[
G_1(x) = \langle O(x) \rangle = \frac{\delta W}{\delta \phi_0(x)}|_{\phi_0=0},
\]

\[
G_2(x) = \langle O(x_1) O(x_2) \rangle_c = \frac{\delta^2 W}{\delta \phi_0(x_1) \delta \phi_0(x_2)}|_{\phi_0=0}.
\]
Now if there is no instability, then $\phi_0$ is small and so is $\phi$, so you can ignore third order terms in $\phi$. From last time:

\[ S[\phi] = \frac{\kappa}{2} \int_{\text{AdS}_{p+2}} d^{p+2}x \sqrt{g} [\phi (-\nabla^2 + m^2) \phi + \mathcal{O}(\phi^3)] - \frac{\kappa}{2} \int_{\partial \text{AdS}} d^{p+1}x \sqrt{\gamma} \phi (n.\partial) \phi, \quad (7) \]

where the last term is the boundary action, $n$ is a normalized vector perpendicular to the boundary and

\[ \nabla^2 = \frac{1}{\sqrt{g}} \partial_A (\sqrt{\gamma} g^{AB} \partial_B). \quad (8) \]

Now if the scalar field satisfies the wave equation:

\[ (-\nabla^2 + m^2)\phi^* = 0, \quad (9) \]

\[ W[\phi_0] = S_{\text{bdy}}[\phi^*[\phi_0]], \quad (10) \]

then we can use translational invariance in $p + 1$ dimensions, $x^\mu \rightarrow x^\mu + a^\mu$, in order to Fourier decompose the scalar field:

\[ \phi(z, x^\mu) = e^{ik.x} f_k(z). \quad (11) \]

Now, substituting (11) into (9) and assuming that the metric only depends on $z$ we get:

\[ 0 = (g^{\mu\nu}k_\mu k_\nu - \frac{1}{\sqrt{g}} \partial_z (\sqrt{\gamma g^{zz} \partial_z}) + m^2) f_k(z) \quad (12) \]

\[ = \frac{1}{L^2} [z^2 k^2 - z^{D+1} \partial_z (z^{-D+1} \partial_z) + m^2 L^2] f_k, \quad (13) \]

where we have used $g^{\mu\nu} = (z/L)^2 \delta^{\mu\nu}$. The solutions of (12) are Bessel functions but we can learn a lot without using their full form. For example, look at the solutions near the boundary (i.e. $z \rightarrow 0$). In this limit we have power law solutions, which are spoiled by the $z^2 k^2$ term. Try using $f_k = z^\Delta$ in (12):

\[ 0 = k^2 z^{2+\Delta} - z^{D+1} \partial_z (\Delta z^{-D+\Delta}) + m^2 L^2 z^\Delta \quad (14) \]

\[ = (k^2 z^2 - \Delta (\Delta - D) + m^2 L^2) z^\Delta, \quad (15) \]

and for $z \rightarrow 0$ we get:

\[ \Delta (\Delta - D) = m^2 L^2 \quad (16) \]

The two roots for (16) are

\[ \Delta_\pm = \frac{D}{2} \pm \sqrt{\left(\frac{D}{2}\right)^2 + m^2 L^2}. \quad (17) \]
Comments

- The solution proportional to $z^{\Delta_-}$ is bigger near $z \to 0$.
- $\Delta_+ > 0 \forall m$, therefore $z^{\Delta_+}$ decays near the boundary.
- $\Delta_+ + \Delta_- = D$.

Next, we want to improve the boundary conditions that allow solutions, so take:

$$\phi(x, z) |_{z=\epsilon} = \phi_0(x, \epsilon) = \epsilon^{\Delta_-} \phi_{0,\text{Ren}}(x),$$

where $\phi_{0,\text{Ren}}$ is the renormalized field. Now with this boundary condition, $\phi(z, x)$ is finite when $\epsilon \to 0$, since $\phi_{0,\text{Ren}}$ is finite in this limit.

Wavefunction renormalization of $O$ (Heuristic but useful)

Suppose:

$$S_{bdy} \ni \int_{z=\epsilon} d^{p+1}x \sqrt{\gamma} \phi_0(x, \epsilon) O(x, \epsilon)$$

$$= \int d^Dx \left( \frac{L}{\epsilon} \right)^D (\epsilon^{\Delta_-} \phi_{0,\text{Ren}}(x)) O(x, \epsilon),$$

where we have used $\sqrt{\gamma} = (L/\epsilon)^D$. Demanding this to be finite as $\epsilon \to 0$ we get:

$$O(x, \epsilon) \sim \epsilon^{D-\Delta_-} O_{\text{Ren}}(x)$$

$$= \epsilon^{\Delta_+} O_{\text{Ren}}(x),$$

where in the last line we have used $\Delta_+ + \Delta_- = D$. Therefore, the scaling of $O_{\text{Ren}}$ is $\Delta_+ \equiv \Delta$.

Comments

- We will soon see that $\langle O(x) O(0) \rangle \sim \frac{1}{|x|^{2\Delta}}$.

- We had a second order ODE, therefore we need two conditions in order to determine a solution (for each $k$). So far we have imposed:
  1. For $z \to \epsilon$, $\phi \sim z^{\Delta_-} \phi_0 + \text{(terms subleading in } z)$. Now we will also impose
  2. $\phi$ regular in the interior of AdS (i.e. at $z \to \infty$).

Comments on $\Delta$

1. The $\epsilon^{\Delta_-}$ factor is independent of $k$ and $x$, which is a consequence of a local QFT (this fails in exotic examples).
2. Relevantness: Since $m^2 > 0 \implies \Delta \equiv \Delta_\pm > D$, so $O_\Delta$ is an irrelevant operator. This means that if you perturb the CFT by adding $O_\Delta$ to the Lagrangian, then:

$$\Delta S = \int d^D x \ (\text{mass})^{D-\Delta} O_\Delta,$$

where the exponent is negative, so the effects of such an operator go away in the IR. For example, consider a dilaton mode with $l > 0$, its mass is given by (for $D = 4$):

$$m^2 = \frac{(l + 4)\bar{l}}{L^2}.$$  

(24)

The operator corresponding to this is:

$$\text{tr}(F^2 X^{i_1 \ldots i_l}),$$

(25)

with $\Delta = 4 + l > D$, therefore it is an irrelevant operator. Now consider a dilaton mode with $l = 0$: then $m^2 = 0$, therefore, $\Delta = D$ and hence it corresponds to a marginal operator (an example of such operator is the Lagrangian). If $m^2 < 0$, then $\Delta < D$, so it corresponds to a relevant operator, but it is ok if $m^2$ is not too negative ("Breitenlohner - Freedman (BF) - allowed tachyons" with $-|m_{BF}|^2 \equiv -(D/2)^2 < m^2$).

3. Instability: This occurs when a renormalizable mode grows with time without a source. But in order to have $S[\phi] < \infty$, the solution must fall off at the boundary. This requires a gradient energy that $\sim \frac{1}{z}$. Note:

$$\Delta_\pm = \frac{D}{2} \pm \sqrt{\left(\frac{D}{2}\right)^2 + m^2 L^2}.$$  

(26)

If:

$$m^2 L^2 < \left(\frac{D}{2}\right)^2 \equiv -|m_{BF}|^2,$$

(27)

then $\Delta_\pm$ is complex, therefore we have $\Delta_- = D/2$, which is larger than the unitary bound. In this case, $\phi \sim z^{\Delta_-}$ decays near the boundary (i.e. in the UV). In order to see the instability that occurs when $m^2 L^2 < \left(\frac{D}{2}\right)^2$ more explicitly, rewrite (9) as a Schrodinger equation, by writing $\phi(z) = A(z) \psi(z)$, where we choose $A(z)$ in order to remove the first derivative of $\psi(z)$. Then, equation (9) becomes:

$$(-\partial_z^2 + V(z))\psi(z) = E\psi(z),$$

(28)

where $E = \omega^2 - k^2$, $V(z) = \sigma/z^2$ and $\sigma = m^2 L^2 - (D^2 - 1)/4$. An instability occurs when $E < 0$, i.e. $\omega^2 < 0$ and hence $\phi \sim e^{i\omega t} \phi(z) = e^{+|\omega|t} \phi(z)$ grows with time. Now the claim is that $V = \sigma/z^2$ has no negative energy states if $\sigma > -1/4$. Note that the notion of normalizability here and before are related (Pset 4):

$$||\psi||^2 = \int dz \psi^\dagger \psi < \infty,$$

(29)

and $S[\phi] = \int dz \sqrt{g} ((\partial \phi)^2 + m^2)$

(30)

4. The formula we found before (expression (16)) depends on the spin. For a $j-$form in AdS we have:

$$\Delta \equiv \Delta_\pm > D,$$

$$m^2 L^2 = (\Delta + j)(\Delta + j - D) = m^2 L^2.$$  

(31)
For example, for $A_\mu$ massless we have:

$$\Delta(j^\mu) = D - 1 \rightarrow \text{conserved}, \quad (32)$$

for $g_{\mu\nu}$ massless we have:

$$\Delta(T^{\mu\nu}) = D \rightarrow \text{required from CFT}. \quad (33)$$